Uvod

$y=f(x)$ je nepoznata funkcija.
pojavljuju se i izvodi te fje pa se zato zove diferencijalna jed.

Pr: $y^2+x\sqrt{y}\cdot y'=0$ dif. jed. 1. reda
$y''-2\operatorname{tg}x\, y'+3y=4-\log x$ 2. reda } najveći izvod u jed.

- Dif. jed. prvog reda možemo dobiti u više oblika:

a) $y'=f(x,y)$ normalni oblik

b) $F(x,y,y')=0$ ne može uvijek se prebacit u a)

c) $P(x,y)dx+Q(x,y)dy=0$ nema izvod ali ima diferencijale (c) uvijek možemo prebaciti u a) i b))

$$Q(x,y)\,dy=-P(x,y)\,dx$$

$$\frac{dy}{dx}=\frac{-P(x,y)}{Q(x,y)}$$

$$y'=-\frac{P(x,y)}{Q(x,y)}\ \rightarrow\ \text{a)}$$

$$Q(x,y)\cdot y'+P(x,y)=0\ \rightarrow\ \text{b)}$$

- Dif. jed. n-tog reda:

a) $y^{(n)}=f(x,y,y',\dots y^{(n-1)})$ normalni oblik

b) $F(x,y,y',\dots y^{(n)})=0$ implicitni oblik

Većina dif. jed. ima ∞ rj., a možemo imati 3 vrste rješenja.

Pr. $y' = x^2$ dif. jed. 1. reda

$y = \int x^2 dx = \frac{x^3}{3} + c$ $c = const$ nam osigurava da jed. ima ∞ rj.

- Opšte rješenje: $y = \varphi(x, c)$
- Partikularno rješenje: ako za c odaberemo neko konkretno rj.

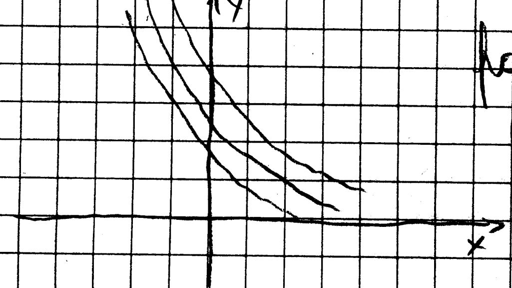
$c = c_1$ - fiksni broj

$\Rightarrow y = \varphi(x, c_1)$

- Singularno rješenje: rj. koje nije sadržano u opštem rj.

Pr.: $y' = y$

Opšte rj.: $y = e^{x+c}$ $(e^{x+c})' = e^{x+c} \Rightarrow y = y'$



familija eksponencijalnih krivih (zavisi od c gdje će presjeći y-osu)

$y = 0 \Rightarrow y' = 0 \Rightarrow y = 0$ je sing. rj. (jer ne možemo dobiti $y = 0$ preko c)

Cauchiyev problem: $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \text{ početni uslov} \end{cases}$

tražimo rj. dif. jed. uz uslov da prođe kroz tačku (x_0, y_0)

Pr. $y' = y$

$y(1) = e^2$

$y = e^{x+c}$ — o.r.

$e^2 = e^{1+c} \Rightarrow c = 1 \Rightarrow y = e^{x+1}$

DIFERENCIJALNE JEDNAČINE PRVOG REDA

Diferencijalne jed. koje razdvajaju promjenljive

$y' = f(x) \cdot g(y) \quad /: g(y),\ g(y) \neq 0$

kasnije se vraćamo da prokomentarišemo da li, ako je $g(y)=0$ imamo singularnih rj. (tj. šta bi bilo da je $g(y)=0$)

$\frac{y'}{g(y)} = f(x) \quad / dx$

$\frac{dy}{g(y)} = f(x)\,dx$

$\Rightarrow \int \frac{dy}{g(y)} = \int f(x)\,dx$

1. $xy\,dx + (x+1)\,dy = 0$ ovdje nam ne treba y'

$xy\,dx = -(x+1)\,dy \quad /: y(x+1), \ (y \neq 0, x \neq -1)$

$\frac{x\,dx}{x+1} = \frac{-dy}{y}$

$\int \frac{x}{x+1}\,dx = -\int \frac{dy}{y}$

$\int \frac{x+1-1}{x+1}\,dx = -\int \frac{dy}{y}$

$\int dx - \int \frac{dx}{x+1} = -\int \frac{dy}{y}$

$x - \ln|x+1| = -\ln|y| + \ln c$

kada imamo logaritme uz konst. c ide log (ln); inače se dodaje c

$x = \ln|x+1| - \ln|y| + \ln c$

$x = \ln \frac{c\,|x+1|}{|y|}$

$\frac{c\cdot(x+1)}{y} = e^{x}$ kada se oslobodimo log. oslobodimo se i apsolutnih vrijednosti

$y = c(x+1)\cdot e^{-x}$ → opšte rj.

Ako je $y=0$ vraćamo se na početak i provjeravamo da li je to rj.:

$y=0 \Rightarrow dy=0$ $\quad 0=0$ — ovo rj. možemo dobiti iz opšteg ako uzmemo $c=0$

$y=0$ → rj. sadržano u opštem

$x=-1$ $\quad dx=0$ $\quad 0=0$ — jeste rj. i nije sadržano u opštem

$x=-1$ je singularno rj.

2. $(x^2-1)y' + 2xy^2 = 0, \quad y(0)=1$

$(x^2-1)y' = -2xy^2 \quad /: y^2(x^2-1), \quad (y\neq 0, x \neq \pm 1)$

obično se ne dijeli sa y^n (prema njemu se orijentiramo rj. na str gdje tu su y-oni)

$\frac{y'}{y^2} = \frac{-2x}{(x^2-1)} \quad / dx$

$\frac{dy}{y^2} = -\frac{2x}{x^2-1}dx$

$\int\frac{dy}{y^2} = -\int\frac{2x}{x^2-1}dx$

$-\frac{1}{y} = -\ln|x^2-1| - c \quad /\cdot(-1)$

$\frac{1}{y} = \ln|x^2-1| + c$

$y\cdot[c+\ln|x^2-1|] = 1$ — o.r.

Ako je: $y=0 \Rightarrow y'=0 \Rightarrow 0=0$ a u o.r. ne možemo dobiti za $y=0$ rj. i pa je:

$y=0$ – singularno r.

$x=\pm 1$ – nisu rj. (to su neke fje) – ne dobijemo identitet

$(x=0 \land y=1) \Rightarrow 1\cdot(c+\underbrace{\ln 1}_{0})=1 \Rightarrow c=1$

pa $c=1$ uvrstimo u opšte rj.

$$y\cdot[1+\ln|x^2-1|]=1$$

$$y=[1+\ln|x^2-1|]^{-1}$$ – traženo partikularno rj.

3. $x^2y'-\cos 2y=1$

$y(+\infty)=\frac{8\pi}{4}$

$$x^2y'=1+\cos 2y$$

$$x^2y'=2\cos^2 y \quad /:2\cos^2 y\cdot x^2$$

$(y\neq\frac{\pi}{2}+k\pi,\ k\in\mathbb{Z},\ x\neq 0)$

$$\frac{y'}{2\cos^2 y}=\frac{1}{x^2} \quad /\cdot dx \quad \text{(integrišemo)}$$

$$\frac{dy}{2\cos^2 y}=\frac{dx}{x^2}$$

$$\int\frac{dy}{2\cos^2 y}=\int\frac{dx}{x^2}$$

$$\frac{1}{2}\operatorname{tg} y=-\frac{1}{x}+C \Rightarrow \boxed{\operatorname{tg} y=-\frac{2}{x}+C_1} \quad (*)$$

Ako je $y=\frac{\pi}{2}+k\pi$ $(k\in\mathbb{Z}) \Rightarrow y$ je neka konst. vrijednost $\Rightarrow$

$y'=0 \Rightarrow 0=0$ (uvrštavanjem u jed.)

$y=\frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}$ – sing. rj.

Ako je $x=0$ nemamo identitet $\Rightarrow$ nije rj.

Još trebamo vidjeti za $y(+\infty)=\frac{9\pi}{4}$:

Ako pustimo u $(*)$ da $x\to+\infty$ i $y=\frac{9\pi}{4}$

$$\operatorname{tg}\frac{9\pi}{4}=C_1 \quad \left(\frac{9\pi}{4}=2\pi+\frac{\pi}{4}\right) \Rightarrow C_1=\operatorname{tg}\frac{\pi}{4}=1$$

$\Rightarrow \operatorname{tg} y=-\frac{2}{x}+1 \to$ partikularno rj.

4. $y'=\cos(y-x)$ ako imamo zbir, razliku x i y ne možemo rastaviti, pa uzimamo smjenu: x ostaje nezavisna promjenljiva, a y uzima smjenu

smjena nepoznate fje.:

$y-x=z,\ z=z(x)$

diferenciranjem ove smjene po x dobijemo:

$$y'-1=z'$$

$$y'=z'+1$$

$$z'+1=\cos z$$

$$z'=-1+\cos z$$

$$z' = -2\sin^2\frac{z}{2} \quad \Big/ : \left(-\sin^2\frac{z}{2}\right)$$

$$\frac{z'}{\sin^2\frac{z}{2}} = 2 \quad \Big/ dx$$

$$\int \frac{dz}{\sin^2\frac{z}{2}} = \int 2dx \qquad \text{smjena } \frac{z}{2} = t \quad z = 2t \quad dz = 2dt$$

$$2\,\text{ctg}\,\frac{z}{2} = 2x + 2c \quad / : 2 \quad (\text{prilagodimo konst.})$$

$$\text{ctg}\,\frac{z}{2} = x + c$$ – opće rj. po z, ali mi vraćamo na x i y

$$\boxed{\text{ctg}\,\frac{y-x}{2} = x + c}$$ – opće rj.

$z = 2k\pi \Rightarrow y - x = 2k\pi, \ k \in \mathbb{Z}$ singularna rj.

Za vježbu:

a) $y'\,\text{ctg}\,x + y = 2$ $\qquad y = 2 - \dfrac{c}{\cos x}$

$y(0) = -1$

b) $e^{-s}\left(1 + \dfrac{ds}{dt}\right) = 1$ $\qquad t = 1 - \dfrac{1}{e^{s}} + c$

c) $y' - xy^2 = 2xy$ $\qquad \ln\dfrac{y}{y+2} = x^2 + c$

d) $xy' + y = y^2$ $\qquad y = \dfrac{1}{1+cx}$

$y(1) = 0.5$

e) $y' = \sqrt{4x + 2y - 1}$ → uvodimo smjenu $z = 4x + 2y - 1$

f) $3y^2 \cdot y' + 16x = 2xy^3$, $y(x)$ je ograničena fja kada $x \to +\infty$

a) $y' \operatorname{ctg} x + y = 2$

$y' \operatorname{ctg} x = 2 - y \quad / : [(2-y)\cdot \operatorname{ctg} x] \qquad (y \neq 2;\ \operatorname{ctg} x \neq 0)$

$\frac{y'}{2-y} = \frac{1}{\operatorname{ctg} x} \quad / dx \qquad x \neq \frac{k\pi}{2} \quad k = 1, 2, \dots$

$\int \frac{dy}{2-y} = \int \frac{dx}{\operatorname{ctg} x} \quad (*)$

$\int \frac{dx}{\operatorname{ctg} x} = \int \frac{1}{\frac{\cos x}{\sin x}} dx = \int \frac{\sin x}{\cos x} dx = \left| (\cos x)' = -\sin x \right| = -\ln|\cos x| + c$

$\frac{1}{y-2} = \frac{c}{\cos x}$

$(*) \ -\ln|y-2| = -\ln|\cos x| + \ln|c| \qquad \ln|y-2|^{-1} = \ln \frac{c}{\cos x} \qquad \boxed{y = 2 - \frac{c}{\cos x}}$ opšte rj.

$y = \frac{\cos x}{c} + 2$

Ako je $y = 2 \Rightarrow 2 = 2$ nije sing. rj. jer ga možemo dobiti iz opšteg ako je $\frac{c}{\cos x} = 0$

Ako je $x = \frac{k\pi}{2} \Rightarrow y' \underbrace{\operatorname{ctg} \frac{k\pi}{2}}_{0} + y = 2 \Rightarrow y = 2$

$x \to 0, \quad y = -1$

$(*) \quad -1 = 2 - \frac{c}{\cos 0°} \qquad -1 - 2 = -c \Rightarrow c = 3$

$y = 2 - \frac{3}{\cos x} \longrightarrow$ partikularno rj.

b) $e^{-s}\left(1 + \frac{ds}{dt}\right) = 1$

$1 + \frac{ds}{dt} = e^{s}$

$\frac{ds}{dt} = e^{s} - 1 \quad / \frac{dt}{e^{s}-1}$

$\int \frac{ds}{e^{s}-1} = \int dt$

$\int \frac{ds}{e^{s}-1} \quad \left| \begin{array}{l} e^{s} = u \\ e^{s} ds = du \end{array} \right| \quad ds = \frac{du}{e^{s}} \quad ds = \frac{du}{u}$

$e^{-s} + e^{-s} \cdot \frac{ds}{dt} = 1 \quad / \cdot dt$

$e^{-s} dt + e^{-s} ds = dt$

$e^{-s} ds = dt - e^{-s} dt$

$e^{-s} ds = (1 - e^{-s}) dt \quad / : (1 - e^{-s}) \qquad 1 - e^{-s} \neq 0 \Rightarrow e^{-s} \neq 1 \quad e^{-s} \neq e^{0} \Rightarrow s \neq 0$

$\int \frac{e^{-s}}{1 - e^{-s}} ds = \int dt$

$\int -\frac{e^{-s} - 1 + 1}{e^{-s} - 1} ds = -\int ds + \int \frac{1}{1 - e^{-s}} ds = -s + \ln|1 - e^{-s}| + c$

$\ln|1 - e^{-s}| - s + c = t \quad (**)$

Ako je $s = 0 \Rightarrow e^{0}(1 + 0) = 1 \Rightarrow 1 = 1$ sing. rj.

a ako je $s = 0$ u $(**) \Rightarrow t = \infty$

c) $y' - xy^2 = 2xy$

$y' = 2xy + xy^2$

$y' = x(2y + y^2) \quad \Big/ : (2y+y^2) \qquad y^2 + 2y \neq 0 \quad y(y+2) \neq 0 \quad y \neq 0 \wedge y \neq -2$

$\frac{y'}{2y+y^2} = x \quad \Big/ dx$

$\int \frac{dy}{2y+y^2} = \int x\,dx$

$y^2 + 2y + 1 - 1 = (y+1)^2 - 1^2$

$\int \frac{dy}{(y+1)^2 - 1^2} = \left| \begin{matrix} y+1 = t \\ dy = dt \end{matrix} \right| = \int \frac{dt}{t^2 - 1} = \frac{1}{2} \ln \left| \frac{t-1}{t+1} \right| = \frac{1}{2} \ln \left| \frac{y}{y+2} \right| + c$

$\frac{1}{2} \ln \left| \frac{y}{y+2} \right| = \frac{x^2}{2} + \frac{c}{2} \quad \Big/ \cdot 2$

$\ln \frac{y}{y+2} = x^2 + c \quad \Big/ e$

$\frac{y}{y+2} = e^{x^2} + k \qquad y = (e^{x^2} + k)\cdot(y+2)$

Ako je: $y = 0 \Rightarrow 0 - 0 = 0$ sing. rj. ✓

$y = -2 \Rightarrow 0 - 4x = -4x$ sing. rj. ✓

d) $xy' + y = y^2 \qquad y(1) = 0.5$

$xy' = y^2 - y \quad \Big/ : (y^2 - y) \qquad y(y-1) \neq 0 \quad y \neq 0 \wedge y \neq 1$

$\frac{xy'}{y^2 - y} = 1 \quad \Big/ : x \qquad x \neq 0$

$\frac{y'}{y^2 - y} = \frac{1}{x} \quad \Big/ dx$

$\frac{1}{y(y-1)} = \frac{a}{y} + \frac{b}{y-1}$

$\int \frac{dy}{y^2 - y} = \int \frac{dx}{x} \qquad -\left(\frac{1}{y} + \frac{1}{y-1}\right) = \frac{y-1-y}{y(y-1)}$

$y^2 - y = y^2 - 2\cdot y \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \left(y - \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2$

$\int \frac{dy}{\left(y-\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} = \left| \begin{matrix} y - \frac{1}{2} = \frac{1}{2}t \\ dy = \frac{1}{2}dt \end{matrix} \right| = \int \frac{\frac{1}{2}dt}{\frac{1}{4}(t^2-1)} = 2 \cdot \int \frac{dt}{t^2-1} = \cancel{2} \cdot \frac{1}{\cancel{2}} \ln \left| \frac{t-1}{t+1} \right| + c$

$t = 2y - 1$

$= \ln \left| \frac{2y-2}{2y} \right| = \ln \left| \frac{y-1}{y} \right| + c$

$\ln\left|\frac{y-1}{y}\right| = \ln|x| + \ln c \qquad \frac{y-1}{y} = c\cdot x \qquad y-1 = cxy$
$\underbrace{\qquad}_{\ln c|x|}$

$y - cxy = 1$

$y(1-cx) = 1$

$y = \frac{1}{1-c\cdot x}$

Ako je: $y=0 \quad 0=0$ sing. rj.

$y = 1 \quad 1 = 1$ sadržana u opštem

za $c=0 \Rightarrow y=1$

$x=1, \; y=0.5$

$0.5 = \frac{1}{1-c\cdot 1} \Rightarrow 0.5 - 0.5\cdot c = 1 \Rightarrow -0.5\cdot c = 0.5 \Rightarrow c = -1$

$y = \frac{1}{1+x} \longrightarrow$ partikularno rj.

e) $y' = \sqrt{4x+2y-1}$

Uvodimo smjenu: $z = 4x+2y-1 \qquad z = z(x)$

$z' = 4 + 2y' \Rightarrow 2y' = z' - 4 \Rightarrow y' = \frac{z'-4}{2} = \frac{z'}{2} - 2$

$\frac{z'}{2} - 2 = \sqrt{z} \qquad \frac{z'}{2} = \sqrt{z} + 2 \quad /\cdot 2$

$z' = 2(2+\sqrt{z}) \quad /:(2+\sqrt{z}) \qquad 2+\sqrt{z} \neq 0 \Rightarrow \sqrt{z} \neq -2 \;/^2 \qquad z \neq 4$

$\frac{z'}{2+\sqrt{z}} = 2 \quad / \int_x dx$

$\int \frac{dz}{2+\sqrt{z}} = \int 2dx \qquad \left| \begin{matrix} z = t^2 \\ dz = 2t\,dt \end{matrix} \right| = \int \frac{2t\,dt}{2+t} = 2\int \frac{t+2}{t+2}dt - 2\int \frac{2}{2+t}dt =$

$= 2t - 4\ln|t+2| + c$

$2\sqrt{z} - 4\ln|\sqrt{z}+2| = 2x + 2c \quad /:2 \qquad = 2\sqrt{z} - 4\ln|\sqrt{z}+2| + c$

$\sqrt{4x+2y-1} - 2\ln\left|\sqrt{4x+2y-1}\right| = x + c$ opšte rj. $\qquad \ln\sqrt{4x+2y-1} = \frac{\sqrt{4x+2y-1} - x - c}{2}$

$4x+2y-1 = e^{\sqrt{4x+2y-1}-x-c}$

$z = 4 \Rightarrow 4x+2y-1 = 4 \qquad y = \frac{5-4x}{2} \qquad y' = -2 \qquad -2 = +2$ sing. rj.

f) $3y^2\cdot y' + 16x = 2xy^3$

$(y-2)(y^2+2y+4)$ $\notin \mathbb{R}$

$3y^2\cdot y' = 2xy^3 - 16x$

$3y^2\cdot y' = 2x(y^3-8) \quad /:(y^3-8) \qquad y^3-8 \neq 0 \Rightarrow y \neq 2$

$\frac{3y^2\cdot y'}{y^3-8} = 2x \quad /dx \qquad \lim_{x\to+\infty} \frac{\sqrt[3]{e^{x^2}+8}}{1} = \lim_{x\to+\infty} \frac{e^{x^2}+8}{\sqrt[3]{(e^{x^2}+8)^2}} \frac{/:e^{x^2}}{/:e^{x^2}} = \lim_{x\to+\infty} \frac{1}{\sqrt[3]{e^{4x^2}}} = 0$

$\int \frac{3y^2}{y^3-8}dy = \int 2x\,dx \qquad$ kada $x \to +\infty \quad y(x) \geq 0$

$\ln|y^3-8| = x^2 + c \Rightarrow y = \sqrt[3]{e^{x^2+c}+8} \Rightarrow y(x)$ je ograničena

$y^3 = e^{x^2+c} + 8$

Homogene diferencijalne jednačine I reda

$y' = f\left(\frac{y}{x}\right)$ nigdje se ne pojavljuje samo x ili samo y osim u razlomku

smjena: $z = \frac{y}{x}$

$\Rightarrow zx = y \quad / \frac{d}{dx}$

$z'x + z = y'$ uvrstimo u početnu:

$\Rightarrow z'x + z = f(z)$

$z'x = f(z) - z$

$\frac{z'}{f(z) - z} = \frac{1}{x} \quad / dx, \int$

$\int \frac{dz}{f(z) - z} = \int \frac{dx}{x}$

/za ovaj tip bitno je da imamo y'/

1. $(x-y)dx + (x+y)dy = 0$ ne mogu se razdvojiti promjenljive

$(x+y)dy = -(x-y)dx$

$\frac{dy}{dx} = -\frac{x-y}{x+y}$

$y' = \frac{-x+y}{x+y} \quad \begin{matrix} /:x \\ /:x \end{matrix}$

$y' = \frac{-1+\frac{y}{x}}{1+\frac{y}{x}} \Rightarrow$ homogena jed.

$\frac{y}{x} = z \Rightarrow y' = z'x + z$

$z'x + z = \frac{-1+z}{1+z}$

$$z'x = \frac{-1+z}{1+z} - z$$

$$z'x = \frac{-1+\cancel{z}-\cancel{z}-z^2}{1+z}$$

$$z'x = -\frac{1+z^2}{1+z} \quad /\cdot \frac{1+z}{1+z^2}\cdot\frac{1}{x} \quad (x \neq 0)$$

ovo je jasno na početku $0+(0+y)dy \neq 0$

$$z'\cdot\frac{1+z}{1+z^2} = -\frac{1}{x} \quad /dx, \int$$

$$\int \frac{1+z}{1+z^2}dz = -\int\frac{dx}{x}$$

$$\int\frac{1}{1+z^2}dz + \frac{1}{2}\int\frac{2z}{1+z^2}dz = -\int\frac{dx}{x}$$

$$\operatorname{arctg} z + \frac{1}{2}\ln(1+z^2) = -\ln|x| + \ln C$$

$$\operatorname{arctg} z = \ln\frac{C}{|x|} - \ln(1+z^2)^{\frac{1}{2}}$$

$$\operatorname{arctg} z = \ln\frac{c}{|x|\cdot\sqrt{1+z^2}} \rightarrow \text{jed. po } z$$

$$z = \frac{y}{x}$$

$$\operatorname{arctg}\frac{y}{x} = \ln\frac{c}{|x|\sqrt{1+\frac{y^2}{x^2}}}$$

$$\operatorname{arctg}\frac{y}{x} = \ln\frac{c}{\sqrt{x^2+y^2}}$$

2. $xy' = y - xe^{\frac{y}{x}} \quad /:x$

$$y' = \frac{y}{x} - e^{\frac{y}{x}}$$

$$\frac{y}{x} = z \Rightarrow y' = z'x + z$$

$$z'x + \cancel{z} = \cancel{z} - e^z$$

$$z'x = -e^z \quad /:x\cdot e^z$$

$$\frac{z'}{e^z} = -\frac{1}{x}$$

$z' \cdot e^{-z} = -\frac{1}{x} \quad / dx, \int$

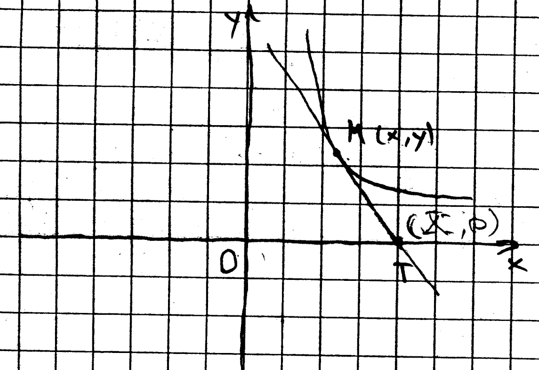
$\int e^{-z} dz = -\int \frac{1}{x} dx$

$-e^{-z} = -\ln|x| - c \quad / \cdot (-1)$

$e^{-z} = \ln|x| + c \quad \rightarrow$ jed. po z $\quad / z = \frac{y}{x} /$

$e^{-\frac{y}{x}} = \ln|x| + c$

3. Naći krive kod kojih je odsječak tangente $\overline{MT}$ od tačke dodira M do presjeka T sa x-osom jednak odsječku $\overline{OT}$ na x-osi, O je ishodište.



$MT = OT$

$Y - y = y'(X - x) \rightarrow$ jednačina tangente; / tražimo koor. T /:

$Y = 0 \Rightarrow -y = y' \cdot X - y'x$

$y' \cdot x - y = y' \cdot X$

$\Rightarrow X = \frac{y'x - y}{y'}$

$X = \frac{y'x}{y'} - \frac{y}{y'}$

$X = x - \frac{y}{y'}$

$T\left(x - \frac{y}{y'}, 0\right)$

krenimo od $\overline{MT} = \overline{OT}$

$$\sqrt{\left(x-\frac{y}{y'}-x\right)^2+(0-y)^2} = \sqrt{\left(x-\frac{y}{y'}\right)^2} \quad /^2$$ udaljenost tačaka

$$\frac{y^2}{(y')^2}+y^2=\left(x-\frac{y}{y'}\right)^2$$

$$\cancel{\frac{y^2}{(y')^2}}+y^2=x^2-2\frac{xy}{y'}+\cancel{\frac{y^2}{(y')^2}}$$

$$y^2-x^2+\frac{2xy}{y'}=0 \quad / \cdot y'$$

$$y'(y^2-x^2)+2xy=0$$

$$y'(y^2-x^2)=-2xy$$

$$y'=-\frac{2xy}{y^2-x^2}$$

$$y'=\frac{2xy \;/ :x^2}{x^2-y^2 \;/ :x^2}$$

$$y'=\frac{\frac{2xy}{x^2}}{1-\frac{y^2}{x^2}}$$

$$y'=\frac{2\cdot\frac{y}{x}}{1-\left(\frac{y}{x}\right)^2} \to \text{homogena}$$

$$\frac{y}{x}=z \Rightarrow y'=z'x+z$$

$$z'x+z=\frac{2z}{1-z^2}$$

$$z'x=\frac{2z}{1-z^2}-z$$

$$z'x=\frac{2z-z+z^3}{1-z^2}$$

$$z'x=\frac{z+z^3}{1-z^2} \quad / \cdot \frac{1-z^2}{z+z^3}\cdot\frac{1}{x} \qquad (z\neq 0)$$

$$z'\cdot\frac{1-z^2}{z+z^3}=\frac{1}{x} \quad / dx, \int$$

$$\int \frac{1-z^2}{z+z^3}\,dz = \int \frac{1}{x}\,dx$$

$$\frac{1-z^2}{z(1+z^2)} = \frac{a}{z} + \frac{bz+c}{1+z^2} \qquad / \cdot z(1+z^2)$$

$$1-z^2 = a(1+z^2) + (bz+c)\cdot z$$

$$1-z^2 = a + az^2 + bz^2 + cz$$

$$a+b=-1$$

$$c=0$$

$$a=1 \Rightarrow b=-2$$

$$\int \frac{1}{z}\,dz - \int \frac{2z\,dz}{1+z^2} = \int \frac{1}{x}\,dx$$

$$\ln|z| - \ln(1+z^2) = \ln|x| + \ln c$$

$$\ln\frac{|z|}{1+z^2} = \ln c|x|$$

$$\frac{z}{1+z^2} = cx$$

$$\frac{\frac{y}{x}}{1+\frac{y^2}{x^2}} = cx$$

$$\frac{\frac{y}{\cancel{x}}}{\frac{x^2+y^2}{x^{\cancel{2}}}} = cx$$

$$\frac{xy}{x^2+y^2} = cx \qquad / : x$$

$$\frac{y}{x^2+y^2} = c$$

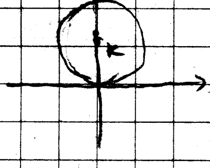
vidimo da su u pitanju kružnice

$$x^2+y^2 = \frac{y}{c}$$

ako stavimo $\frac{1}{c} = 2k$ imamo.

$$x^2+y^2 = 2ky$$

$$x^2+(y-k)^2 = k^2$$



to su kružnice sa centrom na y-osi koje dodiruju x-osu

$$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

$a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{R}$

1° $c_1 = c_2 = 0$

$$y' = \frac{a_1x + b_1y}{a_2x + b_2y} \quad \begin{matrix} /:x \\ /:x \end{matrix}$$

$$y' = \frac{a_1 + b_1\frac{y}{x}}{a_2 + b_2\frac{y}{x}}$$ – homogena uradili smo 1. zad. takođe

2° $$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$$

kad god je ova det. = 0 možemo uzeti ovu smjenu i garantuje nam da ćemo dobiti dif. jed.

$a_1x + b_1y = z$ ili

$a_2x + b_2y = z$

4. $(2x + y + 1)\,dx - (4x + 2y - 3)\,dy = 0$

$(2x + y + 1)\,dx = (4x + 2y - 3)\,dy$

$$\frac{dy}{dx} = \frac{2x + y + 1}{4x + 2y - 3}$$

$$y' = \frac{2x + y + 1}{4x + 2y - 3}$$

$$D = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 4 - 4 = 0$$

$$y' = \frac{2x + y + 1}{2(2x + y) - 3}$$

smjena $2x + y = z$

$$2x + y = z \quad \Big/ \frac{d}{dx}$$

$$2 + y' = z'$$

$$y' = z' - 2$$

$$z' - 2 = \frac{z+1}{2z-3}$$

$$z' = 2 + \frac{z+1}{2z-3}$$

$$z' = \frac{4z - 6 + z + 1}{2z-3}$$

$$z' = \frac{5z-5}{2z-3} \quad \Big/ \cdot \frac{2z-3}{5z-5} \quad (z \neq 1)$$

$$\frac{2z-3}{5(z-1)} \cdot z' = 1 \quad \Big/ dx, \int$$

$$\frac{1}{5} \int \frac{2z-3}{z-1}\, dz = \int dx$$

$$\frac{2z-3}{z-1} = \frac{2z-2}{z-1} - \frac{1}{z-1} = 2 - \frac{1}{z-1}$$

$$\frac{1}{5}\left(2z - \ln|z-1|\right) = x + C \quad \Big/ \cdot 5$$

$$2z - \ln|z-1| = 5x + K \quad \Big/ \underline{K = 5C} \Big/$$

$$2(2x+y) - \ln|2x+y-1| = 5x + K$$

$$0 = \ln|2x+y-1| + x - 2y + K$$

Ako je $z=1$ tj. $2x+y=1$ uvrštavanjem u početnu jed. imamo:

$$\Downarrow$$

$$d(2x+y) = 0$$

$$2dx - (2 \cdot 1 - 3)dy = 0$$

$$2dx + dy = 0$$

$$d(2x+y) = 0$$

Ako uvrstimo $0 \neq \ln|1-1| + \ldots$

$\Rightarrow$ ne spada u o.r.

$2x + y = 1 \Rightarrow y = 1 - 2x$ sing. rj.

Najčešći na ispitu

3°

$$D=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0 \qquad c_1 \neq 0,\ c_2 \neq 0$$

u ovom slučaju uvodimo smjenu i za x i za y

$$x = \underset{\text{novi argument}}{u} + \alpha$$

$$(\alpha, \beta = const)$$

$$y = \underset{\text{nova fja}}{v} + \beta$$

Napomena:

u je novi argument; v je nova fja.

$v = v(u)$

$dx = du,\ dy = dv$

$$\Rightarrow \frac{dy}{dx} = \frac{dv}{du} \Rightarrow y'_x = v'_u$$

$$y' = v' = \frac{a_1(u+\alpha)+b_1(v+\beta)+c_1}{a_2(u+\alpha)+b_2(v+\beta)+c_2}$$

$$v' = \frac{a_1 u + b_1 v + (a_1\alpha + b_1\beta + c_1)}{a_2 u + b_2 v + (a_2\alpha + b_2\beta + c_2)}$$

α i β određujemo iz lin. sis. jed.:

$$\begin{cases} a_1\alpha + b_1\beta + c_1 = 0 \\ a_2\alpha + b_2\beta + c_2 = 0 \end{cases}$$

Determinanta ovog sis. je $D \neq 0$ pa sistem uvijek ima tačno jedno rj.

$$v' = \frac{a_1 u + b_1 v \quad /: u}{a_2 u + b_2 v \quad /: u}$$

$$v' = \frac{a_1 + b_1 \frac{v}{u}}{a_2 + b_2 \frac{v}{u}}$$

$$\frac{v}{u} = z \Rightarrow v' = z + u z' \quad \text{itd.}$$

5. $(2x - 4y + 6)\,dx + (x + y - 3)\,dy = 0$ uvodimo y':

$$(x + y - 3)\,dy = -(2x - 4y + 6)\,dx$$

$$\frac{dy}{dx} = -\frac{2x - 4y + 6}{x + y - 3}$$

$$y' = \frac{-2x + 4y - 6}{x + y - 3}$$

$$D = \begin{vmatrix} -2 & 4 \\ 1 & 1 \end{vmatrix} = -2 - 4 = -6 \neq 0$$

$$\left.\begin{array}{l} x = u + \alpha \\ y = v + \beta \end{array}\right\} y'_x = v'_u$$

$$v' = \frac{-2(u + \alpha) + 4(v + \beta) - 6}{u + \alpha + v + \beta - 3}$$

$$v' = \frac{-2u + 4v + (-2\alpha + 4\beta - 6)}{u + v + (\alpha + \beta - 3)}$$

$$\begin{cases} -2\alpha + 4\beta - 6 = 0 & /: 2 \\ \alpha + \beta - 3 = 0 \end{cases}$$

$$+\begin{cases} -\alpha + 2\beta - 3 = 0 \\ \alpha + \beta - 3 = 0 \end{cases}$$

$3p - 6 = 0$

$\boxed{p = 2}$

$\alpha + 2 - 3 = 0$

$\boxed{\alpha = 1}$

$x = u + 1$
$y = v + 2$ $\Rightarrow$ $\boxed{\begin{matrix} u = x - 1 \\ v = y - 2 \end{matrix}}$ trebat će nam na kraju kada budemo prelazili na x i y

$v' = \dfrac{-2u + 4v}{u + v} \quad \begin{matrix} /:u \\ /:u \end{matrix}$

$v' = \dfrac{-2 + 4 \cdot \frac{v}{u}}{1 + \frac{v}{u}}$ $\qquad \dfrac{v}{u} = z = z(u)$

$v' = z + uz'$

$z + uz' = \dfrac{-2 + 4z}{1 + z}$

$uz' = \dfrac{-2 + 4z - z - z^2}{1 + z}$

$(*)\ uz' = -\dfrac{z^2 - 3z + 2}{1 + z} \quad / : \dfrac{z^2 - 3z + 2}{1 + z} \cdot u \rightarrow (z \neq 1,\ z \neq 2) \quad u \neq 0$

$z' \cdot \dfrac{1 + z}{z^2 - 3z + 2} = -\dfrac{1}{u} \quad / du, \int$

smjenu: V tip u metodi smjene
$a \cdot (2z - 3)$

$\displaystyle\int \frac{1 + z}{(z - 1)(z - 2)}\, dz = -\int \frac{du}{u}$

$\dfrac{1 + z}{(z - 1)(z - 2)} = \dfrac{a}{z - 1} + \dfrac{b}{z - 2} \quad / (z - 1)(z - 2)$

$1 + z = a(z - 2) + b(z - 1)$

$z = 1 \Rightarrow 2 = -a \Rightarrow a = -2$

$z = 2 \Rightarrow 3 = b$

$$\int \frac{-2}{z-1}\,dz + \int \frac{3}{z-2}\,dz = -\int \frac{1}{u}\,du$$

$$-2\ln|z-1| + 3\ln|z-2| = -\ln|u| + \ln C$$

$$-\ln(z-1)^2 + \ln|z-2|^3 = \ln\frac{C}{|u|}$$

$$\ln\left|\frac{(z-2)^3}{(z-1)^2}\right| = \ln\frac{C}{|u|}$$

$$\frac{(z-2)^3}{(z-1)^2} = \frac{C}{u}$$

$$u(z-2)^3 = C(z-1)^2 \qquad z=\frac{v}{u}$$

$$u\left(\frac{v}{u}-2\right)^3 = C\left(\frac{v}{u}-1\right)^2$$

$$u\cdot\left(\frac{v-2u}{u}\right)^3 = C\left(\frac{v-u}{u}\right)^2$$

$$\cancel{u}\frac{(v-2u)^3}{u^{\cancel{3}2}} = C\cdot\frac{(v-u)^2}{u^2} \quad /\cdot u^2$$

$$(v-2u)^3 = C\cdot(v-u)^2 \qquad u=x-1,\; v=y-2$$

$$[y-2-2(x-1)]^3 = C[y-2-(x-1)]^2$$

$$(y-2x)^3 = C(y-x-1)^2$$

Ako je $z=1$; $\frac{v}{u}=1 \Rightarrow v=u$ $\quad /(*)\ \underline{0=0}\ /$

$$\left.\begin{aligned} v&=y-2\\ u&=x-1\end{aligned}\right\} \overset{u=v}{\Longrightarrow} y=x+1$$

$\Downarrow$

sing. rj.

Ako je $z=2 \Rightarrow \frac{v}{u}=2 \Rightarrow v=2u$ $\quad /$ iz $(*) \Rightarrow \underline{0=0}\ /$

$y-2=2(x-1)$ $\qquad y-2=2x-2 \Rightarrow y=2x \rightarrow$ sadržano u opštem. rj.

Za vježbu:

a) $2x^3y' = y(2x^2 - y^2)$ ✓

b) $xy' - y = (x+y)\ln\frac{x+y}{x}$

c) $x - y - 1 + (y - x + 2)y' = 0$

d) $(2x - y + 4)dy + (x - 2y + 5)dx = 0$

e) $xy' = \sqrt{y^2 - x^2} + y$ ✓

a) $2x^3y' = y(2x^2 - y^2)$

$$y' = \frac{2x^2y - y^3}{2x^3} \quad \begin{matrix} /:x^3 \\ /:x^3 \end{matrix}$$

$$y' = \frac{2\cdot\frac{y}{x} - \frac{y^3}{x^3}}{2}$$

$$y' = \frac{y}{x} - \frac{\left(\frac{y}{x}\right)^3}{2}$$

$$z = \frac{y}{x} \qquad y' = z'x + z$$

$$z'x + \cancel{z} = \cancel{z} - \frac{z^3}{2} \quad /\cdot\frac{2}{z^3}\cdot\frac{1}{x} \qquad (z\neq 0, x\neq 0)$$

$$\frac{2}{z^3}\cdot z' = -\frac{1}{x} \quad /dx \int$$

$$\int\frac{2\,dz}{z^3} = -\int\frac{dx}{x}$$

$$2\int z^{-3}dz = -\int\frac{dx}{x}$$

$$2\cdot\frac{z^{-2}}{-2} = -\ln|x| - C$$

$$-\frac{1}{z^2} = -\ln|x| - C \quad /\cdot(-1)$$

$$z^2 = \frac{1}{\ln|x| + C}$$

Ako je $z = 0$, $\frac{y}{x} = 0$ $x \neq 0$ $y = 0$ sing. rj.
$0 = 0$

$$\frac{y^2}{x^2} = \frac{1}{\ln|x| + C}$$

$$y^2 = \frac{x^2}{\ln|x| + C}$$

b) $xy' - y = (x+y)\ln\frac{x+y}{x}$

$$xy' = (x+y)\ln\frac{x+y}{x} + y \quad / : x$$

$$y' = \ln\frac{x+y}{x} + \frac{y}{x}\ln\frac{x+y}{x} + \frac{y}{x}$$

$$y' = \ln\left(1+\frac{y}{x}\right) + \ln\left(1+\frac{y}{x}\right)^{\frac{y}{x}} + \frac{y}{x} \qquad y' - \frac{y}{x} = \ln\left(1+\frac{y}{x}\right) + \frac{y}{x}\ln\left(1+\frac{y}{x}\right)$$

$$y' = \ln\left(1+\frac{y}{x}\right)^{1+\frac{y}{x}} + \frac{y}{x}$$

$$\frac{y}{x} = z \qquad y' = z'x + x$$

$$z'x + z = \ln(1+z)^{1+z} + z$$

$$z'x = \ln(1+z)^{1+z} + \cancel{z} - \cancel{z} \quad / \cdot \frac{1}{x} \cdot \frac{1}{\ln(1+z)^{1+z}} \qquad \boxed{\begin{matrix} z \neq -1 \\ z \neq 0 \end{matrix}}$$

$$\frac{z'}{\ln(1+z)^{1+z}} = \frac{1}{x} \quad / e, dx, \int$$

$$\int \frac{dz}{e^{\ln(1+z)^{1+z}}} = \int \frac{dx}{e^x}$$

$$\int \frac{dz}{(1+z)^{1+z}} = \begin{vmatrix} 1+z = t \\ dz = dt \end{vmatrix} = \int t^{-t}\, dt = \frac{t^{-t+1}}{-t+1} = \frac{(1+z)^{-\cancel{1}-z+\cancel{1}}}{\cancel{1}-\cancel{1}+z} = \frac{1}{z(1+z)^z}$$

$$\frac{1}{z(1+z)^z} = \int e^{-x}\, dx$$

$$\frac{1}{z(1+z)^z} = -e^{-x} + \frac{1}{c}$$

$$\frac{1}{\frac{y}{x}\left(1+\frac{y}{x}\right)^{\frac{y}{x}}} = -\frac{1}{e^x} - \frac{1}{c} \quad /^{-1} \qquad \ln\left(1+\frac{y}{x}\right) = \ln c(x)$$

$$\frac{y}{x}\left(1+\frac{y}{x}\right)^{\frac{y}{x}} + e^x + c = 0$$

Ako je $z = -1 \Rightarrow \frac{y}{x} = -1 \qquad -1(1-1) + e^x + c \neq 0$

$$y = -x$$

c) $x-y-1+(y-x+2)y'=0$

$(y-x+2)y' = y-x+1$

$y' = \dfrac{y-x+1}{y-x+2}$

$D = \begin{vmatrix} -1 & 1 \\ -1 & 1 \end{vmatrix} = -1+1=0$

$-x+y=z \quad \Big/ \dfrac{d}{dx}$

$y'-1=z' \Rightarrow y'=z'+1$

$z'+1 = \dfrac{z+1}{z+2}$

$z' = \dfrac{\cancel{z}+1-\cancel{z}-2}{z+2}$

$z' = \dfrac{-1}{z+2} \quad \Big/ \cdot (z+2) \qquad (z\neq -2)$

$(z+2)\cdot z' = -1 \quad \Big/ dx, \int$

$\int (z+2)\,dz = -\int dx$

$\dfrac{z^2}{2}+2z = -x+\dfrac{C}{2} \quad \Big/ \cdot 2$

$z^2+4z = -2x+C \qquad z = y-x$

$(y-x)^2+4(y-x) = -2x+C$

$y^2-2xy+x^2+4y+2x+C=0$

Ako je $z=-2 \Rightarrow y-x=-2 \Rightarrow y=x+2 \qquad d(y-x)=0$

$\cancel{x}-\cancel{x}+2-1+(\cancel{x}+2-\cancel{x}+2)\cdot 0 = 0 \Rightarrow 1\neq 0$

$$(2x-y+4)\,dy+(x-2y+5)\,dx=0$$

$$(2x-y+4)\,dy=-(x-2y+5)\,dx$$

$$\frac{dy}{dx}=-\frac{x-2y+5}{2x-y+4}$$

$$y'=\frac{-x+2y-5}{2x-y+4}$$

$$D=\begin{vmatrix}-1 & 2\\ 2 & -1\end{vmatrix}=1-4=-3\neq 0$$

$$x=u+\alpha\,,\quad y=v+\beta$$

$$v'=\frac{-1(u+\alpha)+2(v+\beta)-5}{2(u+\alpha)-1(v+\beta)+4}$$

$$v'=\frac{-u-\alpha+2v+2\beta-5}{2u+2\alpha-v-\beta+4}$$

$$v'=\frac{-u+2v+(-\alpha+2\beta-5)}{2u-v+(2\alpha-\beta+4)}$$

$$-\alpha+2\beta-5=0 \quad /\cdot 2$$

$$2\alpha-\beta+4=0$$

$$3\beta-6=0$$

$$\boxed{\beta=2} \Rightarrow v=y-2$$

$$2\alpha-2+4=0$$

$$\boxed{\alpha=-1} \Rightarrow u=x+1$$

$$v'=\frac{-u+2v \quad /:u}{2u-v \quad /:u}$$

$$v'=\frac{-1+2\cdot\frac{v}{u}}{2-\frac{v}{u}} \qquad \frac{v}{u}=z \Rightarrow z'u+z=v'$$

$$z'u+z=\frac{-1+2z}{2-z} \qquad uz'=\frac{-1+\cancel{2z}-\cancel{2z}+z^2}{2-z}$$

$(*)\ uz' = \frac{z^2-1}{2-z} \quad \Big/ \cdot \frac{2-z}{z^2-1} \cdot \frac{1}{u} \qquad z^2 \neq 1 \quad z \neq \pm 1$

$\frac{2-z}{z^2-1} \cdot z' = \frac{1}{u} \quad \Big/ du, \int$

$\int \frac{2-z}{z^2-1} dz = \int \frac{du}{u}$

$\frac{2-z}{(z+1)(z-1)} = \frac{a}{z+1} + \frac{b}{z-1} \quad \Big/ \cdot (z+1)(z-1)$

$2-z = az - a + bz + b$

$2-z = (a+b)z + b - a$

$b - a = 2 \Rightarrow b = 2 + a$

$a + b = -1 \qquad a + 2 + a = -1 \qquad 2a = -3 \qquad a = -\frac{3}{2} \Rightarrow b = \frac{1}{2}$

$\int \frac{-\frac{3}{2}}{z+1} dz + \int \frac{\frac{1}{2}}{z-1} dz = \int \frac{du}{u}$

$-\frac{3}{2} \ln|z+1| + \frac{1}{2} \ln|z-1| = \ln|u| + \ln c$

$\ln \left| \frac{(z-1)^{\frac{1}{2}}}{(z+1)^{\frac{3}{2}}} \right| = \ln c \cdot |u|$

$\frac{(z-1)^{\frac{1}{2}}}{(z+1)^{\frac{3}{2}}} = c \cdot u$

$(z-1)^{\frac{1}{2}} = c \cdot u \cdot (z+1)^{\frac{3}{2}} \qquad \Big/ z = \frac{v}{u} \Big/$

$\left(\frac{v}{u} - 1\right)^{\frac{1}{2}} = c \cdot u \left(\frac{v}{u} + 1\right)^{\frac{3}{2}} \quad \Big/^2$

$\frac{v}{u} - 1 = c^2 \cdot \cancel{u^2} \cdot \frac{(v+u)^3}{u^{\cancel{3}}} \quad \Big/ \cdot u$

$v - u = c^2 (v+u)^3$

$y - 2 - x - 1 = c^2 (y - 2 + x + 1)^3$

$y - x - 3 = c_1 (y + x - 1)^3$

Ako je $z = 1 \quad \frac{v}{u} = 1 \quad \frac{y-2}{x+1} = 1 \quad y - 2 = x + 1 \quad y = x + 3 \qquad 0 = 0 \quad (*)$

$x + 3 - x - 3 \neq c_1 (x + 3 + x - 1)^3 \Rightarrow$ sing. rj.

Ako je $z = -1 \quad v = -u \quad y - 2 = -x - 1 \quad y = -x + 1 \qquad 0 = 0 \quad (*)$

$-x + 1 - x - 3 = c_1 (\cancel{-x} + \cancel{1} + \cancel{x} - \cancel{1})^3 \quad -2 \neq 0$

e) $xy' = \sqrt{y^2 - x^2} + y$

$$y' = \frac{\sqrt{y^2 - x^2} + y}{x}$$

$$y' = \sqrt{\frac{y^2}{x^2} - 1} + \frac{y}{x}$$

$$z = \frac{y}{x} \qquad y' = z'x + z$$

$$z'x + \cancel{z} = \sqrt{z^2 - 1} + \cancel{z} \quad \Big/ \cdot \frac{1}{\sqrt{z^2-1}} \cdot \frac{1}{x}$$

$$\frac{z'}{\sqrt{z^2 - 1}} = \frac{1}{x} \quad \Big/ dx \Big/ \int$$

$$\int \frac{dz}{\sqrt{z^2-1}} = \int \frac{dx}{x}$$

$$\ln \left| z + \sqrt{z^2 - 1} \right| = \ln|x| + \ln c$$

$$\ln \left| z + \sqrt{z^2 - 1} \right| = \ln c \cdot |x|$$

$$z + \sqrt{z^2 - 1} = c \cdot x$$

$$\frac{y}{x} + \sqrt{\frac{y^2}{x^2} - 1} = c x$$

$$\frac{y}{x} + \frac{\sqrt{y^2 - x^2}}{x} = c x \quad \Big/ \cdot x$$

$$y + \sqrt{y^2 - x^2} = c x^2$$

6. $2x^4 y y' + y^4 = 4x^6$

stepeni svakog sabirka trebaju biti isti da bi imali hom. jed.

smjena: $y = z^m$, $m = \text{const.}$

$\Rightarrow y' = m \cdot z^{m-1} \cdot z'$

$$2x^4 \cdot z^m \cdot m \cdot z^{m-1} \cdot z' + (z^m)^4 = 4x^6$$

$$\underbrace{2m\, x^4 \cdot z^{2m-1} \cdot z'}_{} + \underbrace{z^{4m}}_{4m} = \underbrace{4x^6}_{6}$$

ukupni stepen: $2m + 3$

$$2m + 3 = 4m = 6$$

$$2m + 3 = 4m \quad \Big/ - 2m$$

$3 = 2m \Rightarrow m = \frac{3}{2}$

$4m = 6$

$m = \frac{6}{4} \Rightarrow m = \frac{3}{2}$

$y = z^{\frac{3}{2}} \Rightarrow$ jed. je homogena

$3x^4 z^2 \cdot z' + z^6 = 4x^6 \quad / : x^6$

$\frac{3z^2}{x^2} \cdot z' + \frac{z^6}{x^6} = 4$

Smjena: $\frac{z}{x} = u \qquad z = ux \Rightarrow z' = u + xu'$

$3u^2(u + xu') + u^6 = 4$ dobija se jed. koja razdvaja promjenljivu

Za vježbu:

a) $x^3(y' - x) = y^2 \qquad y = x^2\left(1 - \frac{1}{\ln c \cdot |x|}\right)$

b) $2x^2 y' = y^3 + xy \qquad y = x \cdot \sqrt{c} \Rightarrow y = x \cdot c_1$

LINEARNA DIFERENCIJALNA JED. PRVOG REDA

$y' + p(x) \cdot y = g(x)$ ... (1) $\rightarrow$ nehomogena

poznate, nepr. f-je

1° metoda varijacije konstante (Lagrange)

$y' + p(x) \cdot y = 0$ ... (2)

Za jed. (2) kažemo da je homogena lin. dif. jed.

$y' = -p(x) \cdot y$

$\frac{y'}{y} = -p(x) \quad / dx, \int$

$\int \frac{dy}{y} = -\int p(x) dx$

$\ln|y| = -\int p(x) dx + \ln c$

$y = c \cdot e^{-\int p(x) dx}$

Tražimo f-ju $c = c(x)$ tako da je $y = c(x) \cdot e^{-\int p(x) dx}$ rješenje jed. (1).

1. $xy' - 2y = 2x^4 \quad / : x \neq 0$ (y' mora biti samo)

$y' - 2 \cdot \frac{y}{x} = 2x^3$ na mjesto slobodnog člana $2x^3$ stavimo 0

$y' - 2 \cdot \frac{y}{x} = 0$

$y' = 2 \frac{y}{x}$

$\frac{y'}{y} = \frac{2}{x} \quad / dx, \int$

$$\int \frac{dy}{y} = \int \frac{2}{x} dx$$

$$\ln|y| = 2\ln|x| + \ln c$$

$$\ln|y| = \ln c x^2$$

$\underline{y = cx^2}$ → homogena rj.

$$\boxed{y = c(x) \cdot x^2}$$

$$y' = c'(x) \cdot x^2 + c(x) \cdot 2x$$

Uvrstiti u polaznu jed.

$$c'(x) \cdot x^2 + c(x) \cdot 2x - 2 \cdot \frac{c(x) \cdot x^{\cancel{2}}}{\cancel{x}} = 2x^3$$

$$c'(x) \cdot x^2 + \cancel{2c(x)x} - \cancel{2c(x) \cdot x} = 2x^3 \quad / : x^2$$ uvijek se ponište članovi sa $c(x)$

$$c'(x) = 2x$$

$$c(x) = \int 2x \, dx = x^2 + k$$

vratimo sada $c(x)$ u očekivani oblik:

$$y = (x^2 + k) \cdot x^2$$

$$y = x^4 + kx^2$$

2° Metoda nepoznatih funkcija

$$y = uv \Rightarrow y' = u'v + uv'$$

uvrstimo y i y' u (1):

$$u'v + uv' + p(x) \cdot uv = g(x)$$

$$u'v + u\underbrace{(v' + p(x) \cdot v)}_{0} = g(x)$$

radili smo samo sa 1 fjom (y) pa smo je izrazili preko 2 fje (u i v) pa možemo uzeti proizvoljno da je jedna = 0 (u ovom slučaju v)

$$v' + p(x) \cdot v = 0 \Rightarrow v = \ldots$$

$$\Rightarrow u' = \frac{g(x)}{v} \Rightarrow u = \int \frac{g(x)}{v} dx \quad \text{itd.}$$

2. $(xy+e^x)\,dx - x\,dy = 0$ moramo prvo provjeriti da li je jed. linearna

$(xy+e^x)\,dx = x\,dy$ pa posmatramo omjer $\frac{dy}{dx}$:

$\frac{dy}{dx} = \frac{xy+e^x}{x}$ $(x \neq 0)$

$y' = \frac{\cancel{x}y}{\cancel{x}} + \frac{e^x}{x}$

$y' - y = \frac{e^x}{x}$ ⇒ lin. jed.

$y = uv \Rightarrow y' = u'v + uv'$

(3) $u'v + \underbrace{uv' - uv}_{0} = \frac{e^x}{x}$

$u'v + u\underbrace{(v' - v)}_{0} = \frac{e^x}{x}$

$v' - v = 0 \Rightarrow v' = v \Rightarrow \frac{v'}{v} = 1 \quad / dx, \int$

$\int \frac{dv}{v} = \int dx$

$\ln|v| = x$ uzimamo da je $c = 0$

$\Rightarrow v = e^x$

(3) $\Rightarrow u' \cdot e^x = \frac{e^x}{x} \quad / : e^x$

$u' = \frac{1}{x}$

$u = \int \frac{dx}{x} = \ln|x| + C$

$y = e^x(\ln|x| + C)$

Ako je $x = 0 \Rightarrow dx = 0$ i kada uvrstimo u polaznu jed. dobijemo $0 = 0$ ⇒ to je rj. ali da li je s. rj.?
uvrstimo u opšte ⇒ ne možemo dobiti iz opšteg

⇒ $x = 0$ je s. rj.

3° Metoda integracionog faktora

$$y' + p(x)\cdot y = g(x) \qquad \boxed{\cdot e^{\int p(x)dx}} > 0$$

$$y'\cdot e^{\int p(x)dx} + p(x)\cdot y\cdot e^{\int p(x)dx} = g(x) e^{\int p(x)dx}$$

$$\left(y\cdot e^{\int p(x)dx}\right)' = g(x)\cdot e^{\int p(x)dx}$$

$$\Rightarrow y\cdot e^{\int p(x)dx} = \int g(x) e^{\int p(x)dx} dx + C$$

$$y = e^{-\int p(x)dx}\cdot\left(c + \int g(x) e^{\int p(x)dx} dx\right)$$

3. $y = x\,(y' - x\cos x)$

$$y' - x\cos x = \frac{y}{x} \quad (x \neq 0)$$

$$y' - \frac{y}{x} = x\cos x \quad \text{linearna}$$

$$p(x) = -\frac{1}{x}$$

$$\int p(x)dx = -\int \frac{1}{x}dx = -\ln|x| + c$$

uzimamo: $c = 0$ c je proizvoljno

$$e^{\int p(x)dx} = e^{-\ln|x|} = e^{\ln|x^{-1}|} = x^{-1} = \frac{1}{x} \rightarrow \text{integracioni faktor}$$

$$y' - \frac{y}{x} = x\cos x \quad /\cdot\frac{1}{x}$$

nakon množenja integr. faktorom imamo izvod na lijevoj strani

$$y'\cdot\frac{1}{x} - \frac{y}{x^2} = \cos x$$

$$\left(y\cdot\frac{1}{x}\right)' = \cos x$$

$$y\cdot\frac{1}{x} = \int \cos x\, dx$$

$\frac{y}{x} = \sin x + c$

opće rj. $y = x(\sin x + c)$

Ako je $x = 0 \Rightarrow y = 0$
$\downarrow$
nije rj.

ima točke

$\boxed{4}$ $(2e^y - x)\, y' = 1$ pravimo recipročnu vrijednost:

$y' = \frac{1}{2e^y - x}$

$y' = \frac{dy}{dx} \rightarrow \frac{dx}{dy} = 2e^y - x$

$x'(y) = 2e^y - x$

$x' + x = 2e^y \quad / \cdot e^y$ zbog zamjene uloga x i y imamo:

$p(y) = 1 \rightarrow \int p(y)\, dy = y$

$x' e^y + x \cdot e^y = 2e^{2y}$

$(x e^y)'_y = 2e^{2y}$

$x \cdot e^y = \int 2e^{2y}\, dy$

$x \cdot e^y = e^{2y} + c \quad / : e^y$

$x = e^y + \frac{c}{e^y}$

ima trik

5. $y' - \operatorname{tg} y = \frac{e^x}{\cos y}$

$y' - \frac{\sin y}{\cos y} = \frac{e^x}{\cos y} \quad / \cdot \cos y$

$y' \cdot \cos y - \sin y = e^x$

smjena: $\sin y = z \qquad z = z(x)$

$\Rightarrow \cos y \cdot y' = z'$

$z' - z = e^x$ linearna

⋮

Za vježbu:

a) $(2x+1)\,y' = 4x + 2y \qquad y = \ln(2x+1)^{2x+1} + 1) + c \cdot (2x+1)$

b) $x^2 y' + xy + 1 = 0 \qquad y = -\frac{x^3}{4} + \frac{c}{x}$

c) $xy' + (x+1)y = 3x^2 e^{-x} \qquad y = \frac{x^2}{e^x} + \frac{c}{e^x}$

d) $(\sin^2 y + x \operatorname{ctg} y)\,y' = 1 \qquad x = \operatorname{ctg} y \left(\frac{\cos^3 y}{3} - 1\right) + \frac{c}{\sin y}$

e) $(2x+y)\,dy = y\,dx + 4 \ln y\,dy \qquad x = \ln y^2 - y^2 + 1$

f) $(3x - y^2)\,y' = y \qquad x = y^2$

BERNOULLIJEVA DIF. JEDNAČINA

$y' + p(x) \cdot y = g(x) \cdot y^m$ $(m \in \mathbb{R})$ naslanja se na linearnu

kada je $m = 0 \Rightarrow$ linearna

a inače uz pogodnu smjenu se svodi na

lin.; ili direktno - metodama linearnih dif. jed.

Tri načina rješavanja:

1° smjena $y = z^{\frac{1}{1-m}}$ svodi se na lin. d. jed.

2° metoda nepoznatih funkcija

3° metoda varijacije konstante

1. $y' + 2y = y^2 e^x \rightarrow$ Bernulijeva jed.

$m = 2$

smjena: $y = z^{\frac{1}{1-2}} = z^{-1}$

$\Rightarrow y' = -z^{-2} \cdot z'$

$-z^{-2} \cdot z' + 2 \cdot z^{-1} = (z^{-1})^2 \cdot e^x \quad / \cdot (-z^2)$

$z' - 2z = -e^x$

$z = uv \Rightarrow z' = u'v + uv'$

$u'v + uv' - 2uv = -e^x$

$u'v + u \underbrace{(v' - 2v)}_{0} = -e^x$

$v' = 2v$

$\frac{v'}{v} = 2 \quad / dx, \int$

$$\int \frac{dv}{v} = 2\int dx$$

$$\ln|v| = 2x \Rightarrow v = e^{2x}$$

$$u' \cdot e^{2x} = -e^{x} \quad / \cdot e^{-2x}$$

$$u' = -e^{x} \cdot e^{-2x}$$

$$u' = -e^{-x}$$

$$u = \int (-e^{-x})\,dx = e^{-x} + c$$

$$z = e^{2x}(e^{-x} + c)$$

$$z = e^{x} + c \cdot e^{2x}$$

$$y = \frac{1}{z} \Rightarrow z = \frac{1}{y}$$

$$\frac{1}{y} = e^{x} + c \cdot e^{2x}$$

$$\boxed{y(e^{x} + c \cdot e^{2x}) = 1}$$

2. $xy^2 y' = x^2 + y^3 \quad / : xy^2 \quad (x \neq 0, y \neq 0)$

$$y' = \frac{\cancel{x^2}}{\cancel{x}y^2} + \frac{\cancel{y^3}}{x\cancel{y^2}}$$

$$y' = \frac{x}{y^2} + \frac{y}{x}$$

$$y' - \frac{y}{x} = \frac{x}{y^2}$$

$$y = uv \Rightarrow y' = u'v + uv'$$

$$u'v + uv' - \frac{uv}{x} = \frac{x}{u^2v^2}$$

$$(*)\ u'v + u\underbrace{\left(v' - \frac{v}{x}\right)}_{0} = \frac{x}{u^2v^2}$$

$$v' = \frac{v}{x} \Rightarrow \frac{v'}{v} = \frac{1}{x} \quad / \cdot dx, \int$$

$$\int \frac{dv}{v} = \int \frac{1}{x} dx$$

$$\ln|v| = \ln(x)$$

$$v = x$$

$$(*) \Rightarrow u' \cdot x = \frac{\cancel{x}}{u^2 \cdot x^{\cancel{2}}} \quad / \cdot \frac{1}{x} \cdot u^2$$

dobili smo jed. što razdvaja promjenljive pa sve što ima u ide na lijevo, a x desno

$$u' \cdot u^2 = \frac{1}{x^2}$$

$$\int u^2 du = \int x^{-2} dx$$

$$\frac{u^3}{3} = \frac{x^{-1}}{-1} + c$$

$$\frac{u^3}{3} = -\frac{1}{x} + C \quad / \cdot 3$$

$$u^3 = -\frac{3}{x} + k$$ možemo $\sqrt[3]{\ }$ ili y^3:

$$y^3 = u^3 v^3$$

$$y^3 = \left(k - \frac{3}{x}\right) \cdot x^3$$

$$y^3 = kx^3 - 3x^2$$

$x=0$ i $y=0$ nisu rješenja / nije identitet kada uvrstimo u početnu jed.

3. $x\,dx = (x^2 - 2y + 1)\,dy$

$$\frac{dy}{dx} = \frac{x}{x^2 - 2y + 1} \quad /^{-1}$$

$$\frac{dx}{dy} = \frac{x^2 - 2y + 1}{x}$$

$$x' = \frac{x^{\cancel{2}}}{\cancel{x}} - \frac{2y}{x} + \frac{1}{x}$$

$$x' - x = \frac{1 - 2y}{x} \quad \left(\text{tj. } (1-2y) \cdot x^{-1}\right)$$ $n = -1$ Bernulijeva

metoda varijacije konst.:

$$x' - x = 0 \Rightarrow x' = x$$

$$\frac{x'}{x} = 1 \quad / dy, \int$$

$$\int \frac{dx}{x} = \int dy$$

$$\ln|x| = y + \ln c$$

$$x = c \cdot e^y$$

$$x = c(y) \cdot e^y$$

$$x' = c'(y) \cdot e^y + c(y) \cdot e^y$$

$$c'(y) \cdot e^y + \cancel{c(y) e^y} - \cancel{c(y) \cdot e^y} = \frac{1-2y}{c(y) \cdot e^y}$$

$$c'(y) \cdot e^y = \frac{1-2y}{c(y) \cdot e^y} \quad / \cdot e^{-y}$$

$$c'(y) = \frac{1-2y}{c(y) \cdot e^{2y}} \quad / \cdot c(y)$$

$$c(y) \cdot c'(y) = (1-2y) \cdot e^{-2y} \quad / dy, \int \qquad \Big/ \begin{array}{l} c(y) = z \\ c'(y) = z' \end{array} \qquad \int z\, dz = \int c(y) dc(y) \Big/$$

$$\int c(y) \cdot dc(y) = \int (1-2y) e^{-2y} dy$$

$$\frac{[c(y)]^2}{2} = y e^{-2y} + k \quad \text{/parcijalnom integracijom/}$$

$$[c(y)]^2 = 2y e^{-2y} + A$$

$$x^2 = [c(y)]^2 \cdot e^{2y}$$

$$x^2 = (2y e^{-2y} + A) \cdot e^{2y}$$

$$x^2 = 2y + A \cdot e^{2y}$$

unikatan

4. $x\cdot(e^y-y')=2$

$$e^y-y'=\frac{2}{x}$$

$$e^y=y'+\frac{2}{x}$$

zamjena f-je: $e^y=z \Rightarrow y=\ln z \Rightarrow y'=\frac{1}{z}\cdot z'$

$$z=\frac{1}{z}\cdot z'+\frac{2}{x} \quad /\cdot z$$

$$z^2=z'+\frac{2}{x}\cdot z$$

$z'+\frac{2}{x}z=z^2$ Bernulijeva jed. $\Rightarrow$ bilo koja metoda

metoda nepoznatih f-ja:

$$u'v+uv'+\frac{2}{x}\cdot uv=u^2v^2$$

$$u'v+u(\underbrace{v'+\frac{2}{x}v}_{0})=u^2v^2$$

$$v'=-\frac{2}{x}v$$

$$\frac{v'}{v}=\frac{-2}{x} \quad /dx, \int$$

$$\int\frac{dv}{v}=-2\int\frac{dx}{x}$$

$$\ln v=-2\ln x$$

$$\ln v=\ln x^{-2}$$

$$v=x^{-2}=\frac{1}{x^2}$$

$$u'v=u^2v^2 \quad /v$$

$$u'=u^2\cdot v$$

$$u'=u^2\cdot\frac{1}{x^2} \quad /:u^2$$

$$\frac{u'}{u^2}=\frac{1}{x^2}$$

$$\int\frac{du}{u^2}=\int\frac{dx}{x^2} \Rightarrow -\frac{1}{u}=-\frac{1}{x}-c \quad / \cdot(-1)$$

$$\frac{1}{u}=\frac{1}{x}+c$$

$$\frac{1}{u}=\frac{1+cx}{x} \Rightarrow u=\frac{x}{cx+1}$$

$$z=\frac{x}{cx+1}\cdot\frac{1}{x^2}$$

$$e^y=z \Rightarrow e^y=\frac{1}{x(cx+1)}$$

Za vježbu:

a) $(x+1)(y'+y^2)=-y$ $\qquad y=\frac{x^2+3x+2}{x^2}+c$

b) $y'=y^4\cos x+y\,\mathrm{tg}\,x$ $\qquad y=(-3\cos^2 x\cdot\sin x)^{-\frac{1}{3}}$

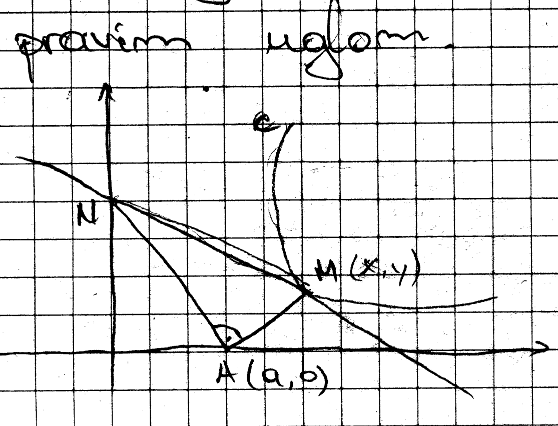
c) $xy'-2x^2\sqrt{y}=4y$ $\qquad y=(\ln|x|+c)^2\cdot x^4$

d) $y'x^3\sin y=xy'-2y$ grupisati $y'=\dots$ $\qquad x=\left(\frac{-y}{\cos y}\right)^{\frac{1}{2}}$

e) $(x^2-1)y'\sin y+2x\cos y=2x-2x^3$ uvodi se smjena $z=\cos y$

$y=\arccos\left[(\ln|x^2-1|+c)\cdot(x^2-1)\right]$

f) Tangenta u proizvoljnoj tački M krivih c siječe y-osu u tački N. Odrediti krive c tako da se odsječci $\overline{MN}$ vide iz tačke $A(a,0)$ pod pravim uglom.



jed. tangente $\Rightarrow$ M

prave AN i AM tj. koeficijente pravca $k\cdot k_1=-1$ (uslov okomitosti)

$$y^2=2ax-a^2 \rightarrow \text{parabola}$$

RICCATI-jeva D.J. I reda

$y' + p(x)\cdot y + Q(x)\cdot y^2 = R(x)$

ako bi $Q(x)=0$ imali bi lin. D.J.

moramo znati jedno partikularno rj.

$y_1 = y_1(x)$ poznato partikularno rj.

smjena: $y = y_1 + \boxed{\frac{1}{z}} \Leftrightarrow z = \frac{1}{y-y_1} \Rightarrow$ lin jed. po fji z.

1. Odrediti konstantu a tako da je $y = \frac{a}{x}$ partikularno rj. date dif. jed. pa zatim rješiti tu jed.

a) $x^2y' + xy + x^2y^2 = 4$

b) $3y' + y^2 + \frac{2}{x^2} = 0$ $a=1: \frac{1}{xy-1} = 1 + \frac{c}{x^{1/3}}$ $a=2: \frac{1}{xy-2} = -1 + c\cdot x^{1/3}$

a) $y = \frac{a}{x} \Rightarrow y' = -\frac{a}{x^2}$

$\cancel{x^2}\cdot\left(-\frac{a}{\cancel{x^2}}\right) + \cancel{x}\cdot\frac{a}{\cancel{x}} + \cancel{x^2}\cdot\frac{a^2}{\cancel{x^2}} = 4$

$\cancel{-a} + \cancel{a} + a^2 = 4 \Rightarrow a = \pm 2$ biramo +2 ili -2 → svejedno!

$a = 2 \Rightarrow y_1 = \frac{2}{x}$

$y = \frac{2}{x} + \frac{1}{z} \Rightarrow y' = -\frac{2}{x^2} - \frac{1}{z^2}\cdot z'$

$x^2\left(-\frac{2}{x^2} - \frac{1}{z^2}\cdot z'\right) + x\cdot\left(\frac{2}{x} + \frac{1}{z}\right) + x^2\left(\frac{2}{x} + \frac{1}{z}\right)^2 = 4$

$-2 - \frac{x^2}{z^2}\cdot z' + 2 + \frac{x}{z} + x^2\left(\frac{4}{x^2} + \frac{4}{xz} + \frac{1}{z^2}\right) = 4$

$-\frac{x^2}{z^2}\cdot z' + \frac{x}{z} + 4 + \frac{4x}{z} + \frac{x^2}{z^2} = \cancel{4} \quad / \cdot (-z^2)\cdot\frac{1}{x^2}$

$z' + \frac{\cancel{x}}{\cancel{z}}\cdot\frac{(-z^2)}{x^2} + 4\cdot\frac{x}{z}\cdot\frac{(-z^2)}{x^2} + \frac{x^2}{\cancel{z^2}}\cdot\frac{-\cancel{z^2}}{x^2} = 0$

$z' - 5 \cdot \frac{z}{x} = 1$ lin. jed.

metoda integracionog faktora

$p(x) = -\frac{5}{x} \Rightarrow \int p(x)dx = -5\ln x$

$e^{\int p(x)dx} = e^{-5\ln x} = e^{\ln x^{-5}} = x^{-5}$

$z' - 5 \cdot \frac{z}{x} = 1 \quad / \cdot x^{-5}$

$x^{-5} \cdot z' - 5z \cdot x^{-6} = x^{-5}$

$(x^{-5} \cdot z)' = x^{-5}$

$x^{-5} \cdot z = \int x^{-5} dx = \frac{x^{-4}}{-4} + C \quad / \cdot x^{5}$

$\boxed{z = -\frac{x}{4} + Cx^{5}}$

$\frac{1}{z} = y - \frac{2}{x}$

$\frac{1}{z} = \frac{xy-2}{x} \Rightarrow \boxed{z = \frac{x}{xy-2}}$

opće rj.: $\frac{x}{xy-2} = -\frac{x}{4} + Cx^{5} \quad / : x$ (jer $x \neq 0$)

$\frac{1}{xy-2} = -\frac{1}{4} + Cx^{4}$

$+a \quad a=-2$

$\frac{1}{xy+2} = \frac{1}{2} + Cx^{2}$

2. Riješiti sljedeće dif. jed. ako se zna da im je jedno rj. oblika $y_1 = ax + b$

a) $y' - 2xy + y^2 = 5 - x^2$

b) $xy' - (2x+1)y + y^2 = -x^2$ $\qquad \frac{1}{y-x-1} = Cx - 1$

c) $y' - y^2 - xy - x + 1 = 0$ $\qquad \frac{1}{y+1} = \frac{1}{2-x} + \frac{C}{e^{\frac{x^2+4x}{2}}}$

d) $(x^3-1)y' = 2xy^2 - x^2y - 1$ $\qquad \frac{1}{y-x} = \frac{x^2+C}{1-x^3}$

a) $y = ax + b \Rightarrow y' = a$

$a - 2x(ax+b) + (ax+b)^2 = 5 - x^2$

$a - 2a x^2 - 2bx + a^2x^2 + 2abx + b^2 = 5 - x^2$

$a^2 - 2a = -1$ (uz x^2)

$-2b + 2ab = 0$ (uz x)

$a + b^2 = 5.$ (slobodni član)

$a^2 - 2a + 1 = 0 \Rightarrow (a-1)^2 = 0 \Rightarrow a = 1$

$2b(a-1) = 0 \Rightarrow b = 0 \vee a = 1$ biramo da bi bile sve 3 jed. zadovoljene

$a + b^2 = 5 \quad b^2 = 5 - 1 \Rightarrow b = \pm 2$

$a = 1, \ b = 2 \Rightarrow y_1 = x + 2$

Smjena: $y = x + 2 + \frac{1}{z} \qquad \frac{1}{z} = y - x - 2$

$y' = 1 - \frac{1}{z^2} \cdot z'$

$1 - \frac{1}{z^2} \cdot z' - 2x\left(x + 2 + \frac{1}{z}\right) + \left(x + 2 + \frac{1}{z}\right)^2 = 5 - x^2$

$\cancel{1} - \frac{z'}{z^2} - \cancel{2x^2} - \cancel{4x} - \cancel{\frac{2x}{z}} + \cancel{x^2} + \cancel{4} + \frac{1}{z^2} + \cancel{4x} + \frac{4}{z} + \cancel{\frac{2x}{z}} = \cancel{5} - \cancel{x^2}$

$-\frac{z'}{z^2} + \frac{1}{z^2} + \frac{4}{z} = 0 \quad / \cdot (-z^2)$

$z' - 1 - 4z = 0$

$z' - 4z = 1$ jed. koja razdvaja promjenljive

$z' = 4z + 1$

$\frac{z'}{4z+1} = 1 \quad / dx, \int$

$\int \frac{dz}{4z+1} = \int dx$

$\frac{1}{4} \ln|4z+1| = x + c$

$\frac{1}{4} \ln\left|4 \cdot \frac{1}{y-x-2} + 1\right| = x + c \quad / \cdot 4$

$\ln\left|\frac{4}{y-x-2} + 1\right| = 4x + k$

EGZAKTNA DIF. JED. I REDA (D.J. SA TOTALNIM DIFERENCIJALOM)

$P(x,y)\,dx + Q(x,y)\,dy = 0$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow du = \underbrace{P(x,y)}_{\frac{\partial u}{\partial x}}dx + \underbrace{Q(x,y)}_{\frac{\partial u}{\partial y}}dy$

$du = 0 \Rightarrow \boxed{u = c}$ opšte rj.

1. $\underbrace{2xy}_{P}\,dx + \underbrace{(x^2 - y^2)}_{Q}\,dy = 0$

$\left.\begin{aligned} \frac{\partial P}{\partial y} &= 2x \\ \frac{\partial Q}{\partial x} &= 2x \end{aligned}\right\} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$\frac{\partial u}{\partial x} = \overbrace{2xy}^{P}$

$\frac{\partial u}{\partial y} = \overbrace{x^2 - y^2}^{Q}$

u zavisi od x i y, integriramo po dx pa nam ostaje $\varphi(y)$

$u = \int 2xy\,dx = y\int 2x\,dx = x^2y + \varphi(y)$

$\Rightarrow \frac{\partial u}{\partial y} = x^2 + \varphi'(y)$

$\Rightarrow \cancel{x^2} + \varphi'(y) = \cancel{x^2} - y^2$

$\varphi'(y) = -y^2 \Rightarrow \varphi(y) = -\int y^2\,dy$

$\varphi(y) = -\frac{y^3}{3} + k$

$u(x,y) = x^2y - \frac{y^3}{3} + k$

opće rj.: $x^2y - \frac{y^3}{3} = c$

II način

$2xy\,dx + (x^2 - y^2)dy = 0$

$y \cdot d(x^2) + x^2 dy - y^2 dy = 0$

$d(uv) = u\,dv + v\,du$ $\qquad \int y^2 dy = \frac{y^3}{3}$

$d(x^2 \cdot y) - d\left(\frac{y^3}{3}\right) = 0$

$d\left(x^2 y - \frac{y^3}{3}\right) = 0$ ovo vrijedi ako je fja = c

$x^2 y - \frac{y^3}{3} = c$

2. $y\,dx - (4x^2 y + x)\,dy = 0 \quad / : x^2$

$\frac{y\,dx}{x^2} - \left(4y + \frac{x}{x^2}\right)dy = 0 \qquad \frac{\partial P}{\partial y} = \frac{1}{x^2} \qquad \frac{\partial Q}{\partial x} = \frac{1}{x^2}$

$\frac{y\,dx}{x^2} - 4y\,dy - \frac{x\,dy}{x^2} = 0$

$\frac{y\,dx - x\,dy}{x^2} - (2y^2)'\,dy = 0 \quad / \cdot (-1)$

diferencijal od količnika

$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$

$\frac{x\,dy - y\,dx}{x^2} + d(2y^2) = 0$

$d\left(\frac{y}{x}\right) + d(2y^2) = 0$

$d\left(\frac{y}{x} + 2y^2\right) = 0$

$\frac{y}{x} + 2y^2 = c \longrightarrow$ opšte rj.

ako jed. nije egzaktna:

$P(x,y)\,dx + Q(x,y)\,dy = 0 \quad / \cdot \mu = \mu(x,y)$

Neka je $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ nakon množenja sa μ treba postati egzaktna

$\mu P\,dx + \mu Q\,dy = 0$

$$\frac{\partial \mu(P)}{\partial y} = \frac{\partial(\mu Q)}{\partial x}$$

$$\mu \cdot \frac{\partial P}{\partial y} + P \cdot \frac{\partial \mu}{\partial y} = \mu \cdot \frac{\partial Q}{\partial x} + Q \cdot \frac{\partial \mu}{\partial x}$$

$$\mu\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q \cdot \frac{\partial \mu}{\partial x} - P \cdot \frac{\partial \mu}{\partial y} \quad \ldots (*)$$

1° Pretpostavimo da je $\mu = \mu(x)$

$$\frac{\partial \mu}{\partial y} = 0, \quad \frac{\partial \mu}{\partial x} = \frac{d\mu}{dx}$$

$$(*) \Rightarrow \mu\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q \cdot \frac{d\mu}{dx}$$

$$\Rightarrow \frac{d\mu}{\mu} = \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q}\,dx \quad \ldots (1)$$

Ova jed. se može iskoristiti ukoliko je razlomak na desnoj strani f-ja koja zavisi samo od x.

2° Pretpostavimo da je $\mu = \mu(y)$

$$\Rightarrow \frac{\partial \mu}{\partial x} = 0 \qquad \frac{\partial \mu}{\partial y} = \frac{d\mu}{dy}$$

$$(*) \Rightarrow \mu\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = -P\frac{d\mu}{dy}$$

$$\Rightarrow \frac{d\mu}{\mu} = \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{-P}\,dy \ldots (2)$$ razdvaja promjenljive

Ako je razlomak na desnoj str. f-ja koja zavisi samo od y, jed. se može iskoristiti.

3. Riješiti sljedeće D.J. ako se zna da imaju integracioni množilac oblika $\mu=\mu(x)$ ili $\mu=\mu(y)$

a) $\underbrace{(x^2+y^2+x)}_{P}dx+\underbrace{y}_{Q}dy=0$

$$\left.\begin{array}{l}\frac{\partial P}{\partial y}=2y\\ \frac{\partial Q}{\partial x}=0\end{array}\right\} \Rightarrow \frac{\partial P}{\partial y}\neq\frac{\partial Q}{\partial x} \Rightarrow \frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}=2y$$

Odredimo da li je $\mu=\mu(x)$ ili $\mu=\mu(y)$. Kada podijelimo sa P i Q vidimo šta nam odgovara.

$\frac{d\mu}{\mu}=\frac{2y}{y}dx$ (iz (1))

$\int\frac{d\mu}{\mu}=2\int dx$

$\ln\mu=2x \Rightarrow \mu=e^{2x}$

$(x^2+y^2+x)dx+y\,dy=0 \quad /\cdot e^{2x}$

$\underbrace{(x^2+y^2+x)e^{2x}}_{P_1}dx+\underbrace{ye^{2x}}_{Q_1}dy=0$

$\frac{\partial P_1}{\partial y}=2ye^{2x}$

$\frac{\partial Q_1}{\partial x}=y\cdot e^{2x}\cdot 2=2ye^{2x}=\frac{\partial P_1}{\partial y}$

$$u=\int(x^2+y^2+x)e^{2x}dx=\left|\begin{array}{ll}U=x^2+y^2+x & dV=e^{2x}dx\\ dU=(2x+1)dx & V=\frac{1}{2}e^{2x}\end{array}\right|=$$

$= (x^2+y^2+x)\cdot\frac{1}{2}e^{2x} - \frac{1}{2}\underbrace{\int (2x+1)e^{2x}\,dx}_{I}$

$$I = \left| \begin{matrix} U = 2x+1 & dV = e^{2x}dx \\ dU = 2dx & V = \frac{1}{2}e^{2x} \end{matrix} \right| = \frac{1}{2}(2x+1)e^{2x} - \int e^{2x}dx =$$

$$= \frac{2x+1}{2}e^{2x} - \frac{1}{2}e^{2x} + k = xe^{2x} + k$$

$$u = (x^2+y^2+x)\cdot\frac{1}{2}e^{2x} - \frac{1}{2}xe^{2x} + \varphi(y)$$

$$= \frac{1}{2}(x^2+y^2)e^{2x} + \varphi(y)$$

$$\Rightarrow \frac{\partial u}{\partial y} = ye^{2x} + \varphi'(y)$$

$$\Rightarrow \cancel{ye^{2x}} + \varphi'(y) = \cancel{ye^{2x}}$$

$$\varphi'(y) = 0 \Rightarrow \varphi(y) = k$$

$$u = \frac{x^2+y^2}{2}e^{2x} + k$$

$\frac{x^2+y^2}{2}e^{2x} = c$ opšte rj.

b) $y\,dy = (x\,dy + y\,dx)\sqrt{1+y^2}$

$$y\,dy - x\sqrt{1+y^2}\,dy - y\sqrt{1+y^2}\,dx = 0 \quad / \cdot(-1)$$

$$\underbrace{y\sqrt{1+y^2}}_{P}\,dx + \underbrace{(x\sqrt{1+y^2} - y)}_{Q}\,dy = 0$$

$$\frac{\partial P}{\partial y} = (y\sqrt{1+y^2})' = \sqrt{1+y^2} + y\cdot\frac{2y}{2\sqrt{1+y^2}} = \sqrt{1+y^2} + \frac{y^2}{\sqrt{1+y^2}}$$

$$\frac{\partial Q}{\partial x} = \sqrt{1+y^2}$$

$$\Rightarrow \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = \frac{y^2}{\sqrt{1+y^2}}$$

postavit ćemo jed (2)

$(2) \Rightarrow \frac{d\mu}{\mu} = \frac{\frac{\cancel{y^2}}{\sqrt{1+y^2}}}{\cancel{+xy}\sqrt{1+y^2}} dy \quad / \int$

$$\int \frac{d\mu}{\mu} = \int \frac{-y}{1+y^2} dy$$

$$\ln|\mu| = -\frac{1}{2}\ln|1+y^2|$$

$$\ln|\mu| = \ln|1+y^2|^{-1/2}$$

$$\mu = (1+y^2)^{-\frac{1}{2}} = \frac{1}{\sqrt{1+y^2}} \rightarrow \text{integracioni množioc}$$

$$y\sqrt{1+y^2}\,dx + (x\sqrt{1+y^2} - y)\,dy = 0 \quad / \cdot \frac{1}{\sqrt{1+y^2}}$$

$$\underbrace{y}_{P_1}\,dx + \underbrace{\left(x - \frac{y}{\sqrt{1+y^2}}\right)}_{Q_1} dy = 0$$

$$\frac{\partial P_1}{\partial y} = 1 = \frac{\partial Q_1}{\partial x} \Rightarrow \text{jed. totalnog diferencijala}$$

* za vježbu preko fje μ - na standardni način

$$y\,dy = (x\,dy + y\,dx)\sqrt{1+y^2} \quad / \cdot \frac{1}{\sqrt{1+y^2}}$$

$$\frac{y\,dy}{\sqrt{1+y^2}} = \underbrace{x\,dy + y\,dx}_{\text{proizvod diferencijala}}$$

$$d\left(\sqrt{1+y^2}\right) = d(xy)$$

$$\sqrt{1+y^2} = xy + C$$

c) $(2x^2y + 2y + 5)\,dx + (2x^3 + 2x)\,dy = 0 \qquad 2xy + 5\,\mathrm{arctg}\,x = c$

d) $2x\,\mathrm{tg}\,y\,dx + (x^2 - 2\sin y)\,dy = 0 \qquad x^2\sin y + \cos^2 y = c$

e) $xy^2(xy' + y) = 1 \qquad$ Uputa: $y' = \frac{dy}{dx}$ pa $/ \cdot dx$

$$\frac{x^3y^3}{3} - \frac{x^2}{2} = c$$

4. Riješiti dif. jed.:

$(2x-y)dx+(x+2y)dy=0$ ako se zna da ona ima integracioni množilac oblika $\mu=\mu(x^2+y^2)$

$$\mu\left(\underbrace{\frac{\partial P}{\partial y}}_{-1}-\underbrace{\frac{\partial Q}{\partial x}}_{1}\right)=Q\cdot\frac{\partial\mu}{\partial x}-P\cdot\frac{\partial\mu}{\partial y}$$

$$\mu=\mu(\omega),\qquad \omega=x^2+y^2$$

$$\frac{\partial\mu}{\partial x}=\mu'_\omega\cdot\frac{\partial\omega}{\partial x}=\mu'\cdot 2x$$

$$\frac{\partial\mu}{\partial y}=\mu'_\omega\cdot\frac{\partial\omega}{\partial y}=\mu'\cdot 2y$$

$$\mu(-1-1)=(x+2y)\cdot\mu'\cdot 2x-(2x-y)\cdot\mu'\cdot 2y$$

$$-2\mu=\mu'\cdot 2\cdot(x^2+\cancel{2xy}-\cancel{2xy}+y^2)\quad /:2$$

$$-\mu=\mu'(x^2+y^2)$$

$$-\mu=\frac{d\mu}{d\omega}\cdot\omega$$

$$\frac{d\mu}{\mu}=-\frac{d\omega}{\omega}$$

$$\int\frac{d\mu}{\mu}=-\int\frac{d\omega}{\omega}$$

$$\ln|\mu|=-\ln|\omega|$$

$$\ln|\mu|=\ln|\omega|^{-1}$$

$$\mu=\frac{1}{\omega}=\frac{1}{x^2+y^2}$$

vraćamo se na početnu jed. i $/ \cdot \frac{1}{x^2+y^2}$

$$(2x-y)\,dx + (x+2y)\,dy = 0 \quad / \cdot \frac{1}{x^2+y^2}$$

$$\underbrace{\frac{2x-y}{x^2+y^2}}_{P_1} dx + \underbrace{\frac{x+2y}{x^2+y^2}}_{Q_1} dy = 0$$

$$\frac{\partial P_1}{\partial y} = \frac{-1\cdot(x^2+y^2)-(2x-y)\cdot 2y}{(x^2+y^2)^2} = \frac{-x^2-y^2-4xy+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2-4xy}{(x^2+y^2)^2}$$

$$\frac{\partial Q_1}{\partial x} = \frac{x^2+y^2-(x+2y)\cdot 2x}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2-4xy}{(x^2+y^2)^2} = \frac{y^2-x^2-4xy}{(x^2+y^2)^2}$$

polaznu jed. ćemo grupisati: /ili preko fje u/

$$\frac{2x\,dx}{x^2+y^2} - \frac{y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2} + \frac{2y\,dy}{x^2+y^2} = 0$$

$$\frac{2x\,dx+2y\,dy}{x^2+y^2} + \frac{x\,dy-y\,dx}{x^2+y^2}\,\begin{matrix}/:x^2\\ /:x^2\end{matrix} = 0$$

a da je $y\,dx-x\,dy$ dijelimo sa y^2

$$\frac{d(x^2+y^2)}{x^2+y^2} + \frac{\frac{x\,dy-y\,dx}{x^2}}{1+\frac{y^2}{x^2}} = 0$$

$$\int\frac{dt}{t} = \ln|t|$$

$$t = x^2+y^2$$

$$d(\ln(x^2+y^2)) + \frac{d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2} = 0$$

$$d(\ln(x^2+y^2)) + d\left(\operatorname{arc\,tg}\frac{y}{x}\right) = 0$$

$$\ln(x^2+y^2) + \operatorname{arc\,tg}\frac{y}{x} = c$$

5. Riješiti sljedeće jednačine, ako se zna da im je integracioni množilac oblika $\mu = \mu(x^{\alpha} y^{\beta})$ $\alpha, \beta \in \mathbb{R}$

a) $(6xy^2 + x^2)y' - y(3y^2 - x) = 0$

b) $x \cdot (y^2 - 3x)y' + 2y^3 - 5xy = 0$

c) $(2x^{\frac{5}{2}}y^{\frac{3}{2}} + x^2 y - x)y' - x^{\frac{3}{2}}y^{\frac{5}{2}} + xy^2 - y = 0$

nema diferencijala, pa ih moramo uvesti

b) $\frac{1}{x^{25}y^{13}}\left(\frac{1}{5y^2} - \frac{1}{17x}\right) = C$

c) $\frac{-x^2}{2} + y^2 - 2x^{-\frac{1}{2}}y^{-\frac{1}{2}} + \frac{2}{3}x^{-\frac{3}{2}}y^{\frac{3}{2}} = C$

a) $(6xy^2 + x^2)\frac{dy}{dx} - y(3y^2 - x) = 0 \quad / \cdot dx$

$$(6xy^2 + x^2)\,dy - (3y^3 - xy)\,dx = 0$$

$$\underbrace{(-3y^3 + xy)}_{P}dx + \underbrace{(6xy^2 + x^2)}_{Q}dy = 0$$

$$\frac{\partial P}{\partial y} = -9y^2 + x \qquad \frac{\partial Q}{\partial x} = 6y^2 + 2x$$

krenimo od opšte relacije za μ:

$$\mu\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q \cdot \frac{\partial \mu}{\partial x} - P \cdot \frac{\partial \mu}{\partial y}$$

$$\mu = \mu(\omega) \qquad \omega = x^{\alpha} y^{\beta}$$

$$\frac{\partial \mu}{\partial x} = \mu'_{\omega} \cdot \omega'_x = \mu'_{\omega} \cdot \alpha x^{\alpha - 1} y^{\beta}$$

$$\frac{\partial \mu}{\partial y} = \mu'_{\omega} \cdot \omega'_y = \mu'_{\omega} \cdot x^{\alpha} \cdot \beta y^{\beta - 1}$$

sada sve uvrstimo u glavnu relaciju:

$$\mu(-9y^2 + x - 6y^2 - 2x) = (6xy^2 + x^2)\mu' \alpha x^{\alpha-1} y^{\beta} - (-3y^3 + xy)\mu' x^{\alpha} \beta y^{\beta-1}$$

$$\mu(-15y^2 - x) = \mu'(6\alpha x^{\alpha} y^{\beta+2} + \alpha x^{\alpha+1} y^{\beta} + 3\beta x^{\alpha} y^{\beta+2} - \beta x^{\alpha+1} y^{\beta})$$

$$\mu(-15y^2 - x) = \frac{d\mu}{d\omega}\left[(6\alpha + 3\beta)x^{\alpha} y^{\beta+2} + (\alpha - \beta)x^{\alpha+1} y^{\beta}\right]$$

$$\frac{d\mu}{\mu} = \frac{-15y^2 - x}{(6\alpha + 3\beta)x^{\alpha} y^{\beta+2} + (\alpha - \beta)x^{\alpha+1} y^{\beta}}\,d\omega$$

veći stepen uz y je $y^{\beta+2}$ izjednačimo sa $-15y^2$:

$6\alpha + 3\beta = -15$

$\alpha - \beta = -1 \quad / \cdot 3$

$$\left.\begin{aligned} 6\alpha + 3\beta &= -15 \\ 3\alpha - 3\beta &= -3 \end{aligned}\right\} +$$

$9\alpha = -18 \qquad \alpha = -2$

$-2 - \beta = -1 \qquad -2 + 1 = \beta \Rightarrow \beta = -1$

$$\frac{d\mu}{\mu} = \frac{-15y^2 - x}{-15x^{-2}y - x^{-1}} \, dw$$

$$-15x^{-2}y - x^{-1}y^{-1} = -\frac{15y}{x^2} - \frac{1}{xy} = \frac{-15y^2 - x}{x^2 y}$$

sada se vratimo na jed:

$$\frac{d\mu}{\mu} = \frac{\cancel{-15y^2 - x}}{\frac{\cancel{-15y^2 - x}}{x^2 y}} \, dw$$

$$\frac{d\mu}{\mu} = x^2 y \, dw$$

$w = x^{\alpha} y^{\beta} = x^{-2} y^{-1} \qquad w^{-1} = x^2 y$

$$\frac{d\mu}{\mu} = w^{-1} \, dw$$

$$\int \frac{d\mu}{\mu} = \int \frac{dw}{w}$$

$$\ln \mu = \ln w$$

$\mu = w = \frac{1}{x^2 y} \Rightarrow$ integracioni množilac

vratimo se na sređeni oblik polazne jed:

$(-3y^3 + xy)\,dx + (6xy^2 + x^2)\,dy = 0 \quad / \cdot \frac{1}{x^2 y}$

$$\underbrace{\left(-\frac{3y^2}{x^2} + \frac{1}{x}\right)}_{P_1} dx + \underbrace{\left(\frac{6y}{x} + \frac{1}{y}\right)}_{Q_1} dy = 0$$

$$\frac{\partial P_1}{\partial y} = -\frac{6y}{x^2}$$
$$\frac{\partial Q_1}{\partial x} = -\frac{6y}{x^2}$$

provjera da li su jednaki ⇒ jesu ⇒ egzaktna jed.

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{6y}{x}dy - \frac{3y^2}{x^2}dx = 0$$

$$d(\ln x) + d(\ln y) + \frac{6xy\,dy - 3y^2\,dx}{x^2} = 0$$

$$d(\ln x + \ln y) + d\left(\frac{3y^2}{x}\right) = 0$$

$$\ln x + \ln y + \frac{3y^2}{x} = c$$

2) a) $(x - xy)\,dx + (x^2 + y)\,dy = 0$

$\mu = \mu(x^2 + y^2)$ $\qquad \dfrac{y-1}{\sqrt{x^2+y^2}} = c$

b) $(2x^2y^3 - 1)y\,dx + (4x^2y^3 - 1)x\,dy = 0$

$\mu = \mu(xy)$ $\qquad 2xy + \dfrac{1}{xy} = c$

c) $(x^2 + x^2y + 2xy - y^2 - y^3)\,dx + (y^2 + xy^2 + 2xy - x^2 - x^3)\,dy = 0$

$\mu = \mu(x + y)$

JEDNAČINE KOJE NISU RIJEŠENE PO y'

I Metoda: Rješavanje po y'

1. $8(y')^3 = 27y$

$$(y')^3 = \frac{27y}{8}$$

$$y' = \sqrt[3]{\frac{27y}{8}}$$

$$y' = \frac{3}{2}\sqrt[3]{y} \quad / : \sqrt[3]{y} \quad (y \neq 0)$$

$$\frac{y'}{\sqrt[3]{y}} = \frac{3}{2} \quad / dx, \int$$

$$\int \frac{dy}{\sqrt[3]{y}} = \frac{3}{2}\int dx$$

$$\int y^{-\frac{1}{3}} dy = \frac{y^{\frac{2}{3}}}{\frac{2}{3}} + c = \frac{3}{2}\sqrt[3]{y^2} + c$$

$$\frac{3}{2}\sqrt[3]{y^2} = \frac{3}{2}x + \frac{3}{2}c \quad / : \frac{3}{2}$$

$$\sqrt[3]{y^2} = x + c \Rightarrow \boxed{y^2 = (x+c)^3} \text{ - opšte rj.}$$

Ako je $y = 0 \Rightarrow y' = 0 \Rightarrow 0 = 0$; ne može se dobiti iz opšteg rj. $\Rightarrow$ $y = 0$ je singularno rj.

2. $(y')^2 + xy = y^2 + xy'$ → D.J. koja se razdvaja na dvije kada se implicitno izrazi

$(y')^2 - xy' + xy - y^2 = 0$

$y' = t$

$\underset{a=1}{t^2} \underbrace{-xt}_{b} + \underbrace{xy - y^2}_{c} = 0$

$D = x^2 - 4(xy - y^2) = x^2 - 4xy + 4y^2 = (x-2y)^2$

$t_{1,2} = \frac{x \pm \sqrt{(x-2y)^2}}{2} = \frac{x \pm (x-2y)}{2}$

$t_1 = \frac{x + x - 2y}{2} = x - y \qquad t_2 = \frac{x - x + 2y}{2} = y$

$y' = x - y$ ili $y' = y$

$y' + y = x \quad / \cdot e^x$ ~~ili $x - y = z$~~

$e^x y' + e^x y = xe^x$

$(e^x y)' = xe^x$

$e^x y = \int xe^x dx = e^x(x-1) + C_1 \quad /: e^x$

$\boxed{y = x - 1 + \frac{C_1}{e^x}}$ 1. opšte

$y' = y \quad /: y \quad y \neq 0$

$\frac{y'}{y} = 1 \quad / dx, \int$

$\int \frac{dy}{y} = \int dx$

$\ln|y| = x + C_2$

$\boxed{y = e^{x + C_2}}$ 2. opšte

$y = 0$ → sing. rj.

Za vježbu:

a) $(y')^2 = 4y^3$ $\quad y = \frac{1}{(C_1 - x)^2}$ opšte; $y = 0$ sing.

b) $x(y')^2 = y$ $\quad y = (\sqrt{x} + C_1)^2$ opšte; $y = 0$ sing.

c) $y(y')^3 + x = 1$: $\quad y = \sqrt[3]{(C_1 - (1-x)^4)^4}$

d) $xy'(xy' + y) = 2y^2$ $\quad y = \frac{c^2}{x^2} \wedge y = cx$

e) $(y')^2 - 2xy' = 8x^2$ $\quad y = -x^2 + c \wedge y = 2x^2 + c$

f) $x \cdot (y - xy')^2 = x(y')^2 - 2yy'$ $\quad y = (1 + \sqrt{1-x^2})c \wedge y = \frac{1-\sqrt{1-x^2}}{x^2} \cdot c$

II Metoda: uvođenje parametra

kada x ili y zavise od y'

1. $x = (y')^3 + y'$

$\boxed{y' = p} \Rightarrow x = p^3 + p$

$dx = (3p^2 + 1)\,dp$

ako uspijemo i y izraziti preko p onda rj možemo izraziti u parametarskom obliku

$\boxed{dy = p\,dx}$

$dy = p(3p^2 + 1)\,dp$

$dy = (3p^3 + p)\,dp$

$y = \int (3p^3 + p)\,dp$

$y = 3\int p^3\,dp + \int p\,dp$

$y = 3 \cdot \frac{p^4}{4} + \frac{p^2}{2} + C$

Opće rješenje:

$$\begin{cases} x = p^3 + p \\ y = \frac{3p^4}{4} + \frac{p^2}{2} + C \end{cases}$$

možemo onda p izraziti pa imamo fju u implicitnom obliku

2. $y=(y')^2+2(y')^3$

$y'=p$

$y=p^2+2p^3$

$dy=(2p+6p^2)\,dp$

$p\,dx=(2p+6p^2)\,dp \quad /:p$

$dx=(2+6p)\,dp$

$x=\int(2+6p)\,dp=2p+6\cdot\frac{p^2}{2}+c$

$x=2p+3p^2+c$

$$R_j:\begin{cases}x=2p+3p^2+c\\ y=p^2+2p^3\end{cases}$$

3. $(y')^2-2xy'=x^2-4y$

$4y=x^2-(y')^2+2xy'$

$y=\frac{x^2-(y')^2+2xy'}{4}$

$y'=p \Rightarrow y=\frac{x^2-p^2+2xp}{4} \quad /d$

$dy=\frac{2x\,dx-2p\,dp+2(x\,dp+p\,dx)}{4}$ umjesto dy pišemo $p\,dx$

$p\,dx=\frac{2x\,dx-2p\,dp+2x\,dp+2p\,dx}{4} \quad /\cdot 4$

$4p\,dx=(2x+2p)\,dx+(2x-2p)\,dp$

$(4p-2x-2p)\,dx=(2x-2p)\,dp$

$(2p-2x)\,dx=(2x-2p)\,dp \quad /:2$

$(p-x)\,dx=(x-p)\,dp$

$-(x-p)\,dx=(x-p)\,dp$

ne smijemo odma podijeliti nego postavimo pretpostavke:

1° $x \neq p$ podijelimo sa $(x-p)$

$-dx = dp \Rightarrow dx = -dp$

$\int dx = -\int dp \Rightarrow x = -p + C$

$x = -p + C \Rightarrow p = c - x$

$\cancel{y = \frac{x^2 - p^2 + 2xp}{4}}$

$$y = \frac{x^2 - (c-x)^2 + 2x(c-x)}{4}$$

$$y = \frac{\cancel{x^2} - c^2 + 2cx + \cancel{x^2} + 2xc - 2x^2}{4}$$

$$y = \frac{4cx - 2x^2 - c^2}{4} \rightarrow \text{opće rj.}$$

2° $x = p$

$\cancel{y = \frac{x^2 - p^2 + 2xp}{4}}$

u o. rj. uz x^2 je -2 to je $-\frac{1}{2}x^2$ tako da se $\frac{x^2}{2}$ ne može dobiti iz o.r.

$$y = \frac{\cancel{x^2} - \cancel{x^2} + 2x \cdot x}{4} = \frac{x^2}{2} \rightarrow \text{sing. rj.}$$

4. a) $x\left[(y')^2 - 1\right] = 2y'$ $\quad \begin{cases} x = \frac{2p}{p^2-1} \\ y = \ln \frac{1}{(p+1)\sqrt{p^2-1}} + \frac{1}{p-1} + C \end{cases}$

b) $y'(x - \ln y') = 1$ $\quad \begin{cases} x = \frac{1}{p} + \ln p \\ y = p - \ln p + c \end{cases}$

c) $y = \ln\left[1 + (y')^2\right]$ $\quad \begin{cases} x = 2\,\mathrm{arctg}\, p + c \\ y = \ln(1+p^2) \end{cases}$

d) $(y'+1)^3 = (y' - y)^2$ $\quad \begin{cases} x = \ln p - 3\sqrt{p+1} + \frac{3}{2}\ln\left|\frac{1+\sqrt{p+1}}{1-\sqrt{p+1}}\right| + c \\ y = p - (p+1)\sqrt{p+1} \end{cases}$

e) $x^2(y')^2 = xyy' + 1$

f) $2xy' - y = y' \ln(yy')$

Lagranžova (Lagrange) D.J.

$y = x\cdot f(y') + g(y')$

Kleroova (Clairaut) D.J. - specijalan slučaj Lagranžove

$y = xy' + g(y')$

uvodi se parametar $y'=p$ pa dif.

L.D.J. se u opštem slučaju svodi na lin. D.J. po nepoznatoj fji $x = x(p)$ $(p = y')$.
može se svesti i na jed. koja razdvaja prom.

Kod Kleroove jed. može se zaključiti da je $P=C$, (odakle slijedi opšte rj. $y = xC + g(C)$) i imamo još i uslov iz kojeg se eventualno može dobiti singularno rj.

5. a) $y = x\underset{f(y')}{y'} - (y')^2$ => Kleroova

Kod Kleroove ćemo uvijek imati pdx na lijevoj i na desnoj str.

$y' = p \Rightarrow y = xp - p^2 \quad / d$

$dy = x dp + p dx - 2p dp$

$p dx = x dp + p dx - 2p dp$

$0 = dp(x - 2p)$

1° $dp = 0 \Rightarrow p = C$

$\underline{y = xc - c^2}$ - opšte rješenje

2° $x - 2p = 0 \Rightarrow x = 2p \Rightarrow p = \frac{x}{2}$

$y = x\cdot\frac{x}{2} - \left(\frac{x}{2}\right)^2 \Rightarrow y = \frac{x^2}{4}$ - sing. rj.

b) $y = x\underbrace{y'(y'+2)}_{f(y')}$ $\Rightarrow$ Lagranžova, $g(y')=0$

$y'=p \Rightarrow y = xp(p+2)$

$$\boxed{y = xp^2 + 2xp} \quad /d$$

$$dy = x\cdot 2p\,dp + p^2 dx + 2(x\,dp + p\,dx)$$

$$p\,dx = (p^2+2p)dx + (2xp+2x)dp$$

$$(p-p^2-2p)dx = 2x(p+1)dp$$

$$(-p^2-p)\,dx = 2x(p+1)dp$$

$$-p(p+1)dx = 2x(p+1)dp$$

1° $p \neq 0,\ p \neq -1$ $\quad (-p(p+1) \neq 0)$

$-p\,dx = 2x\,dp \quad /:px \quad$ ($x \neq 0$ u startu)

$$\int -\frac{dx}{x} = \int \frac{2\,dp}{p}$$

$$-\ln|x| = 2\ln|p| + \ln c$$

$$\ln|x^{-1}| = \ln p^2\cdot c$$

$$\frac{1}{x} = p^2\cdot c \Rightarrow p^2 = \frac{1}{cx} \Rightarrow p = \pm\frac{1}{\sqrt{cx}}$$

Kada smo izrazili p vraćamo se na početak:

$$y = x\cdot\frac{1}{cx} + 2x\cdot\left(\frac{\pm 1}{\sqrt{cx}}\right)$$

$$y = \frac{1}{c} \pm 2\sqrt{\frac{x}{c}}$$

$\frac{1}{c} = k \Rightarrow \boxed{y = k \pm 2\sqrt{kx}}\ (*)$ opšte rj.

2° $p=0$

$y = xp^2 + 2xp \Rightarrow y = 0$ možemo dobiti iz $(*)$ rješ.

3° $p=-1 \Rightarrow y = x - 2x$

$y = -x$ – sing. rj. $\quad$ x je pod korijenom u $(*)$ pa ne možemo dobiti $y = -k \pm 2$ $(*)$

c) $y = 2xy' + \ln y'$ Lagranžova

$y' = p \Rightarrow y = 2xp + \ln p \quad / d$

$$dy = 2(x\,dp + p\,dx) + \frac{1}{p}dp$$

$$p\,dx = 2p\,dx + \left(2x + \frac{1}{p}\right)dp$$

$$(p - 2p)\,dx = \left(2x + \frac{1}{p}\right)dp$$

$$-p\,dx = \left(2x + \frac{1}{p}\right)dp$$

većinom se dobija jed. kod koje se ne mogu razdvajati promjenljive, ona se svodi na lin.

1° $p \neq 0$

$$\frac{dx}{dp} = \frac{2x + \frac{1}{p}}{-p}$$

$$x_p' = \frac{2x}{-p} + \frac{\frac{1}{p}}{-p}$$

$x' + \frac{2x}{p} = -\frac{1}{p^2}$ → lin. D.J. po x koja zavisi od p

$x = uv \quad x' = u'v + uv'$

$$u'v + uv' + \frac{2}{p}uv = -\frac{1}{p^2}$$

$$u'v + u\left(v' + \frac{2}{p}v\right) = -\frac{1}{p^2}$$

$$v' + \frac{2}{p}v = 0$$

$$v' = -\frac{2}{p}v \quad / : v$$

$$\frac{v'}{v} = -\frac{2}{p} \quad / dp, \int$$

$$\int \frac{dv}{v} = \int \frac{-2\,dp}{p}$$

$$\ln|v| = -2\ln|p|$$

$$v = p^{-2} = \frac{1}{p^2}$$

$u' \cdot \frac{1}{p^2} = -\frac{1}{p^2} \quad / p^2$

$u' = -1 \quad / dp, \int$

$u = \int dp = -p + c$

$x = \frac{c-p}{p^2}$

$y = 2xp + \ln p$

$y = 2p \cdot \frac{c-p}{p^2} + \ln p$

$y = \frac{2(c-p)}{p} + \ln p$

Opće rj.:

$$\begin{cases} x = \frac{c-p}{p^2} \\ y = \frac{2(c-p)}{p} + \ln p \end{cases}$$

3° $p = 0$

Jed nije definisana

Za vježbu:

6. a) $y + xy' = 4\sqrt{y'}$

b) $y = 2xy' - 4(y')^3$

c) $y = xy' - (2 + y')$

d) $(y')^3 = 3(xy' - y)$ → Kleroova; izraziti y ⇒ Kleroova

e) Naći krive kojima tangenta u proizvoljnoj tački T polovi ugao između ordinate tačke i prave koja spaja tačku T sa ishodištem.

RAZNI PRIMJERI D.J.-NA 1. REDA

1. $(x^2+1)y' + x\sin y\cos y - x(x^2+1)\cos^2 y = 0 \quad /:\cos^2 y \qquad y \neq \frac{\pi}{2} + k\pi$

$$(x^2+1)\frac{y'}{\cos^2 y} + \frac{x\sin y\cancel{\cos y}}{\cos^2 y} - x(x^2+1) = 0$$

$$(x^2+1)(\operatorname{tg} y)' + x\operatorname{tg} y - x(x^2+1) = 0$$

smjena: $\operatorname{tg} y = z$

$$(x^2+1)z' + xz - x(x^2+1) = 0 \quad /:(x^2+1)$$

$$z' + \frac{x}{x^2+1}z - x = 0 \rightarrow \text{lin.}$$

$$\int \frac{x}{x^2+1}dx = \frac{1}{2}\int\frac{2x}{x^2+1}dx = \frac{1}{2}\ln(x^2+1)$$

$$e^{\frac{1}{2}\ln(x^2+1)} = e^{\ln(x^2+1)^{\frac{1}{2}}} = (x^2+1)^{\frac{1}{2}} = \sqrt{x^2+1}$$

$$z'\sqrt{x^2+1} + \frac{x\,z}{x^2+1}\sqrt{x^2+1} = x\sqrt{x^2+1}$$

$$\left(z\sqrt{x^2+1}\right)' = x\sqrt{x^2+1}$$

$$z\sqrt{x^2+1} = \int x\sqrt{x^2+1}\,dx = \left| \begin{array}{l} x^2+1 = t^2 \\ x\,dx = t\,dt \end{array} \right| = \int t^2 dt = \frac{t^3}{3} + c = \frac{(\sqrt{x^2+1})^3}{3} + c$$

$$z = \frac{(\sqrt{x^2+1})^2}{3} + \frac{c}{\sqrt{x^2+1}}$$

$$z = \frac{x^2+1}{3} + \frac{c}{\sqrt{x^2+1}}$$

$$z = \operatorname{tg} y \Rightarrow \operatorname{tg} y = \frac{x^2+1}{3} + \frac{c}{\sqrt{x^2+1}} \rightarrow \text{opće rj.}$$

$$y = \frac{\pi}{2} + k\pi \;(k\in\mathbb{Z}) \Rightarrow y' = 0$$

$$y = \frac{\pi}{2} + k\pi \;(k\in\mathbb{Z}) \rightarrow \text{sing. rj.}$$

2. $y' + y^2 - 2x^2 y + x^4 - 2x - 1 = 0$

$y' + (y - x^2)^2 - 2x - 1 = 0$

$y - x^2 = z \qquad y = z + x^2 \qquad y' = z' + 2x$

$z' + 2x + z^2 - 2x - 1 = 0$

$z' = 1 - z^2 \quad / : (1 - z^2) \qquad z \neq \pm 1$

$\frac{z'}{1 - z^2} = 1 \quad / dx, \int$

$\int \frac{dx}{1 - z^2} = \int dx$

$\frac{1}{2} \ln \left| \frac{1+z}{1-z} \right| = x + c \quad / \cdot 2$

$\ln \left| \frac{1+z}{1-z} \right| = 2x + k, \qquad k = 2c$

$\ln \left| \frac{1 + y - x^2}{1 - y + x^2} \right| = 2x + k$

$z = \pm 1 \Rightarrow z' = 0$

$y - x^2 = \pm 1$

$y = x^2 \pm 1 \rightarrow$ sing. rješenje

3. $y' = \frac{y - x^2 \sqrt{x^2 - y^2}}{xy\sqrt{x^2 - y^2} + x}$ nije homogena ali ćemo uzeti smjenu za hom.:

$\frac{y}{x} = u \Rightarrow y = ux \Rightarrow y' = u'x + u \qquad u' = \frac{du}{dx}$

$u'x + u = \frac{ux - x^2 \sqrt{x^2 - u^2 x^2}}{x^2 u \sqrt{x^2 - u^2 x^2} + x}$

$u'x + u = \frac{ux - x^3 \sqrt{1 - u^2}}{x^3 u \sqrt{1 - u^2} + x}$

$u'x + u = \frac{\cancel{x}(u - x^2 \sqrt{1 - u^2})}{\cancel{x}(x^2 u \sqrt{1 - u^2} + 1)}$

$$u'x+u=\frac{u-x^2\sqrt{1-u^2}}{x^2u\sqrt{1-u^2}+1}$$

$$u'x=\frac{u-x^2\sqrt{1-u^2}}{x^2u\sqrt{1-u^2}+1}-u \quad / : x\neq 0$$

$$u'=\frac{u-x^2\sqrt{1-u^2}}{x^3u\sqrt{1-u^2}+x}-\frac{u}{x} \qquad u'=\frac{du}{dx}$$

$$u'=\frac{\cancel{u}-x^2\sqrt{1-u^2}-u^2x^2\sqrt{1-u^2}-\cancel{u}}{x(x^2u\sqrt{1-u^2}+1)}$$

$$u'=\frac{-x^{\cancel{2}}\sqrt{1-u^2}(1+u^2)}{\cancel{x}(x^2u\sqrt{1-u^2}+1)}$$

$$\frac{du}{dx}=\frac{-x\sqrt{1-u^2}(1+u^2)}{x^2u\sqrt{1-u^2}+1}$$

$$(ux^2\sqrt{1-u^2}+1)\,du=-x\sqrt{1-u^2}(1+u^2)\,dx$$

$$x\sqrt{1-u^2}(1+u^2)\,dx+(ux^2\sqrt{1-u^2}+1)\,du=0 \quad / : \sqrt{1-u^2}$$

$$\underbrace{x(1+u^2)}_{P}dx+\underbrace{\left(ux^2+\frac{1}{\sqrt{1-u^2}}\right)}_{Q}du=0$$

$$\frac{\partial P}{\partial u}=x\cdot 2u=2ux$$

$$\Rightarrow \text{egzaktna D.J.}$$

$$\frac{\partial Q}{\partial x}=u\cdot 2x=2ux$$

⋮

opće rj. $x^2+y^2+2\cdot\arcsin\frac{y}{x}=C$

Za vježbu:

4. a) $(x^2+y^2+1)yy' + (x^2+y^2-1)x = 0$

b) $(x^2+y^2+x)y' - y = 0$

(smjena: $y = ux$)

c) $(2x^2y+x)y' - x^2y^3 + 2xy^2 + y = 0$

d) $[(y')^2+1]\sin^2(xy'-y) = 1$

Uputa: riješiti po y $\Rightarrow$ Kleroova

e) $6x^5y\,dx + (y^4\ln y - 3x^6)\,dy = 0$

uzeti neki integracioni množilac ili smjer $\frac{dy}{dx}$ ili $\frac{dx}{dy}$

D.J. VIŠEG REDA

* Snižavanje reda diferencijalne jednačine

- Metoda uzastopnog integriranja
- Uvođenje smjene
- Homogene diferencijalne jed. višeg reda
- Metoda namještanja izvoda na obje strane

I. $y^{(n)} = f(x)$, $n \in \mathbb{N}$

$n \geq 2 \Rightarrow y^{(n-1)} = \int f(x)dx$: uzastopnim integriranjem na lijevoj str. dobićemo y

1. a) $y'' = xe^x$

$$y' = \int xe^x dx = \begin{vmatrix} u = x & dv = e^x dx \\ du = dx & v = e^x \end{vmatrix} = xe^x - \int e^x dx = xe^x - e^x + A$$

$$y = \int (xe^x - e^x + A)\,dx = \int xe^x dx - \int e^x dx + \int A\,dx =$$

$$= xe^x - e^x - e^x + Ax + B$$

$$y = (x-2)\,e^x + Ax + B \qquad (A, B - \text{const.})$$

b) $y''' = \cos^2 x$, $y(0) = 1$, $y'(0) = -\frac{1}{8}$, $y''(0) = 0$

3 početna uslova → da izračunamo 3 const. da dobijemo partikularno rj.

$$y'' = \int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \int \frac{1}{2}dx + \frac{1}{2}\int \cos 2x\,dx =$$

$$= \frac{1}{2}x + \frac{1}{2} \cdot \frac{1}{2}\sin 2x + C_1$$

$$y'' = \frac{1}{2}x + \frac{1}{4}\sin 2x + C_1$$

$$y''(0) = 0 \Rightarrow 0 = C_1$$

$\Rightarrow y'' = \frac{1}{2}x + \frac{1}{4}\sin 2x$

$y' = \int \frac{1}{2}x\,dx + \int \frac{1}{4}\sin 2x\,dx = \frac{1}{2}\cdot\frac{x^2}{2} + \frac{1}{4}\cdot\left(-\frac{1}{2}\right)\cdot\cos 2x + C_2$

$= \frac{x^2}{4} - \frac{1}{8}\cos 2x + C_2$

$y'(0) = -\frac{1}{8} \Rightarrow -\frac{1}{8} + C_2 = -\frac{1}{8} \Rightarrow C_2 = 0$

$\Rightarrow y' = \frac{x^2}{4} - \frac{1}{8}\cos 2x$

$y = \int \frac{x^2}{4}dx - \frac{1}{8}\int \cos 2x\,dx = \frac{1}{4}\cdot\frac{x^3}{3} - \frac{1}{8}\cdot\frac{1}{2}\sin 2x + C_3 =$

$= \frac{x^3}{12} - \frac{1}{16}\sin 2x + C_3$

$y(0) = 1 \Rightarrow 0 - \frac{1}{16}\cdot\sin 0 + C_3 = 1 \Rightarrow C_3 = 1$

$$\boxed{y = \frac{x^3}{12} - \frac{1}{16}\sin 2x + 1}$$

c) $y'' = xe^{-x}$ $\quad y(0) = 1,\ y(1) = -1$

d) $x\cdot y^{IV} = 1$

e) $y'' = 4\sin 2x$ $\quad y(0) = 0,\ y'(0) = 0$

f) $y'''\cdot\sin^4 x = \sin 2x$

II $F(x, y^{(k)}, y^{(k+1)}, \dots, y^{(n)}) = 0$

smjena: $y^{(k)} = z$

$\Rightarrow y^{(k+1)} = z', \; y^{(k+2)} = z'', \dots$

2. a) $x^2 y'' = (y')^2$

$y' = z \Rightarrow y'' = z'$

$x^2 \cdot z' = z^2 \quad / : (x^2 \cdot z^2), \; x \neq 0, \; z \neq 0$

$\frac{z'}{z^2} = \frac{1}{x^2} \quad / dx, \int$

$\int \frac{dz}{z^2} = \int \frac{dx}{x^2}$

$-\frac{1}{z} = -\frac{1}{x} - C \quad / \cdot (-1)$

$\frac{1}{z} = \frac{1}{x} + C$

$\frac{1}{z} = \frac{1+Cx}{x} \Rightarrow z = \frac{x}{1+Cx}$

$z = y' \Rightarrow y' = \frac{x}{1+Cx}$

$y = \int \frac{x}{1+Cx} dx = \frac{1}{C} \int \frac{Cx}{1+Cx} dx = \frac{1}{C} \int \frac{Cx+1-1}{1+Cx} dx =$

$= \frac{1}{C} \int dx - \frac{1}{C} \int \frac{dx}{1+Cx} = \frac{1}{C} x - \frac{1}{C} \ln |Cx+1| + D$

$y = \frac{1}{C} x - \frac{1}{C} \ln |Cx+1| + D$

b) $y'' - x y''' + (y''')^3 = 0$

$y'' = z \Rightarrow y''' = z'$

$z - x z' + (z')^3 = 0$

$z = xz' - (z')^3$) Kleroova

$z' = p \qquad z = px - p^3 \quad / d$

$dz = x\,dp + p\,dx - 3p^2\,dp$

$\cancel{p\,dx} = x\,dp + \cancel{p\,dx} - 3p^2\,dp$

$0 = (x - 3p^2)\,dp$

1° $dp = 0 \Rightarrow p = c \Rightarrow z = xc - c^3$

$y'' = cx - c^3$

$y' = \int (cx - c^3)\,dx = c \cdot \frac{x^2}{2} - c^3 x + A$

$y = \int c \cdot \frac{x^2}{2}\,dx - \int c^3 x\,dx + A\int dx =$

$= \frac{c}{2} \cdot \frac{x^3}{3} - c^3 \cdot \frac{x^2}{2} + Ax + B$

$y = \frac{c}{6} x^3 - \frac{c^3}{2} x^2 + Ax + B$

2° $(x - 3p^2) = 0 \Rightarrow p^2 = \frac{x}{3} \qquad p = \sqrt{\frac{x}{3}}$ ispitati ima li sing. rj.

c) $2xy'y'' = (y')^2 - 1$

d) $y''' - (y')^2 = (y'')^3$

e) $xy'' = y' + x((y')^2 + x^2)$

f) $xy'' = y' + x \sin \frac{y'}{x}$

III $F(y, y', y'') = 0$

smjena: $y' = p$, $p = p(y)$

$$y'' = \frac{d(y')}{dx} = \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = p' \cdot p$$

$$y''' = p'' \cdot p + p' \cdot p' = p \cdot p'' + (p')^2$$

3.a) $y^3 \cdot y'' = 1$

$y' = p(y) \Rightarrow y'' = pp'$

$y^3 \cdot pp' = 1 \quad / : y^3$ jed. koja razdvaja prom.

$pp' = y^{-3} \quad / dy, \int$

$$\int p\,dp = \int y^{-3}\,dy$$

$$\frac{p^2}{2} = \frac{y^{-2}}{-2} + \frac{A}{2} \quad / \cdot 2$$

$$p^2 = -\frac{1}{y^2} + A \Rightarrow p^2 = \frac{Ay^2 - 1}{y^2} \Rightarrow p = \pm \frac{\sqrt{Ay^2 - 1}}{y}$$

$$y' = \pm \frac{\sqrt{Ay^2 - 1}}{y} \quad / \cdot \frac{y}{\sqrt{Ay^2 - 1}}$$

$$\frac{yy'}{\sqrt{Ay^2 - 1}} = \pm 1 \quad / dx, \int$$

$$\int \frac{y\,dy}{\sqrt{Ay^2 - 1}} = \pm \int dx$$

$$\left| Ay^2 - 1 = t^2 \Rightarrow 2Ay\,dy = 2t\,dt \Rightarrow y\,dy = \frac{1}{A} t\,dt \right|$$

$$\frac{1}{A}\int\frac{t\,dt}{t}=\frac{1}{A}t+B=\frac{1}{A}\cdot\sqrt{Ay^2-1}+B$$

$$\frac{1}{A}\cdot\sqrt{Ay^2-1}+B=\pm x \quad /\cdot A$$

$$\boxed{\sqrt{Ay^2-1}+AB=\pm Ax}$$

$$\sqrt{Ay^2-1}=A(\pm x-B) \quad /^2$$

$$Ay^2-1=A^2(x^2\pm 2Bx+B^2)$$

b) $y'' \cos y+(y')^2\sin y=y'$

uz uslove: $y(-1)=\frac{\pi}{6}$ $\quad y'(-1)=2$

$$y'=p(y) \Rightarrow y''=pp'$$

$$pp'\cos y+p^2\sin y=p \quad /:p,\ p\neq 0$$

$$p'\cdot\cos y+p\sin y=1 \quad /:\cos y,\ y\neq\frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}$$

(*)... $p'+p\,tg\,y=\frac{1}{\cos y}$ $\quad$ lin. jed.

$$p'+p\,tg\,y=0$$

$$p'=-p\,tg\,y$$

$$\frac{p'}{p}=-\frac{\sin y}{\cos y} \quad /dy,\int$$

$$\int\frac{dp}{p}=\int-\frac{\sin y}{\cos y}\,dy$$

$$\ln|p|=\ln|\cos y|+\ln c$$

$$p=c\cdot\cos y$$

$$p=c(y)\cos y$$

$p' = c' \cdot \cos y - c(y) \sin y$

$(*) \Rightarrow c' \cdot \cos y \cancel{- c(y) \sin y} + \cancel{c(x) \cos y} \cdot \frac{\sin y}{\cos y} = \frac{1}{\cos y}$

$$c' = \frac{1}{\cos^2 y}$$

$$c = \int \frac{dy}{\cos^2 y} \qquad c = \operatorname{tg} y + A$$

$p = (\operatorname{tg} y + A) \cdot \cos y$

$p = \left(\frac{\sin y}{\cos y} + A \right) \cdot \cos y$

$p = \sin y + A \cos y$

$y' = p \Rightarrow y' = \sin y + A \cos y$ jed. koja razdvaja prom.

Kada je $x = -1 \Rightarrow y = \frac{\pi}{6}$ a $y' = 2$ pa imamo:

$$-2 = \sin \frac{\pi}{6} + A \cos \frac{\pi}{6}$$

$$2 = \frac{1}{2} + A \cdot \frac{\sqrt{3}}{2}$$

$$2 - \frac{1}{2} = A \cdot \frac{\sqrt{3}}{2}$$

$$\frac{3}{2} = A \cdot \frac{\sqrt{3}}{2} \quad / \cdot 2$$

$$3 = A\sqrt{3} \Rightarrow A = \frac{3}{\sqrt{3}} = \frac{\sqrt{9}}{\sqrt{3}} \Rightarrow A = \sqrt{3}$$

$y' = \sin y + \sqrt{3} \cos y$

$\frac{y'}{\sin y + \sqrt{3} \cos y} = 1 \quad / dx, \int$

$$\underbrace{\int \frac{dy}{\sin y + \sqrt{3} \cos y}}_{\text{univerzalna smjena } \operatorname{tg}\frac{x}{2} \text{ ili:}} = \int dx$$

$$a\cos y + b\sin y = \sqrt{a^2+b^2}\left(\underbrace{\frac{a}{\sqrt{a^2+b^2}}}_{\sin\varphi}\cos y + \underbrace{\frac{b}{\sqrt{a^2+b^2}}}_{\cos\varphi}\sin y\right)$$

$$= \sqrt{a^2+b^2}\cdot\sin(y+\varphi)$$

$$a=\sqrt{3} \quad b=1 \Rightarrow \sqrt{a^2+b^2} = \sqrt{3+1} = 2$$

$$\sin y + \sqrt{3}\cos y = 2\left(\frac{1}{2}\sin y + \frac{\sqrt{3}}{2}\cos y\right) =$$

$$= 2\left(\cos\frac{\pi}{3}\cdot\sin y + \sin\frac{\pi}{3}\cos y\right) = 2\cdot\sin\left(y+\frac{\pi}{3}\right)$$

$$\int\frac{dy}{2\sin\left(y+\frac{\pi}{3}\right)} = \int dx$$

$$\left| y+\frac{\pi}{3} = t \Rightarrow dy = dt \right|$$

$$\frac{1}{2}\int\frac{dt}{\sin t} = \frac{1}{2}\ln\left|\operatorname{tg}\frac{t}{2}\right| + B$$

$$\frac{1}{2}\ln\left|\operatorname{tg}\frac{y+\frac{\pi}{3}}{2}\right| = x + B$$

$$x=-1 \Rightarrow y=\frac{\pi}{6}$$

$$\frac{1}{2}\ln\left|\operatorname{tg}\frac{\frac{\pi}{6}+\frac{\pi}{3}}{2}\right| = -1+B$$

$$\underbrace{\frac{1}{2}\ln\left|\operatorname{tg}\frac{\pi}{4}\right|}_{0} = -1+B \quad \Rightarrow B=1$$

$$\frac{1}{2}\ln\left|\operatorname{tg}\left(\frac{y}{2}+\frac{\pi}{6}\right)\right| = x+1 \quad \Big/ \cdot 2$$

$$\ln\left|\operatorname{tg}\left(\frac{y}{2}+\frac{\pi}{6}\right)\right| = 2x+2$$

c) $(y')^2+2yy''=0$

d) $2yy''=y^2+(y')^2$

e) $y''=e^y$

f) $y''+(y')^2=2e^{-y}$

IV $f'(x)=g'(x) \Rightarrow f(x)=g(x)+C \qquad C=const.$

4. a) $y''=xy'+y+1$

$$(y')'=(xy)'+x'$$

$$(y')'=(xy+x)'$$

$$y'=xy+x+A$$

$$y'-xy=x+A \quad \text{linearna D.J.} \quad /\cdot e^{-\frac{x^2}{2}}$$

$$\int -x\,dx=-\frac{x^2}{2}$$

$$y'e^{-\frac{x^2}{2}}-xye^{-\frac{x^2}{2}}=e^{-\frac{x^2}{2}}(x+A)$$

$$\left(ye^{-\frac{x^2}{2}}\right)'=e^{-\frac{x^2}{2}}(x+A)$$

$$ye^{-\frac{x^2}{2}}=\underbrace{\int xe^{-\frac{x^2}{2}}dx}_{I}+\underbrace{\int Ae^{-\frac{x^2}{2}}dx}=$$

int. koji se ne može integrirati u konačnom obliku pa ga ostavljamo kao f-ju

$$I=\int xe^{-\frac{x^2}{2}}dx=\left|\begin{matrix}-\frac{x^2}{2}=t\\ -x\,dx=dt\end{matrix}\right|=\int -e^{t}dt=-e^{t}+B=-e^{-\frac{x^2}{2}}+B$$

$$ye^{-\frac{x^2}{2}}=-e^{-\frac{x^2}{2}}+B+\int Ae^{-\frac{x^2}{2}}dx \quad /\cdot e^{\frac{x^2}{2}}$$

$$y=1+B\cdot e^{\frac{x^2}{2}}+Ae^{\frac{x^2}{2}}\int e^{-\frac{x^2}{2}}dx$$

b) $y y'' = y'(y'+1)$ uklapa se u ovaj oblik kada nemamo x

$$y y'' = (y')^2 + y'$$

$$y y'' - (y')^2 = y' \quad /: y^2$$

$$\frac{y y'' - (y')^2}{y^2} = \frac{y'}{y^2}$$

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$\left(\frac{y'}{y}\right)' = \left(\frac{-1}{y}\right)'$$

$$\frac{y'}{y} = -\frac{1}{y} + A \quad / \cdot y$$

$$y' = -1 + Ay$$ jed. koja razdvaja prom.

c) $y' \cdot y''' = 2(y'')^2$

d) $y \cdot y'' + 3y' \cdot y'' = 0$

Uputa: $/: (y'')^2$

e) $x y'' = 2 y y' - y'$

f) $x y'' - y' = x^2 y y'$

5. Koristiti osobinu homogenosti D.J. višeg reda:

a) $y y'' = (y')^2 + 15 y^2 \sqrt{x}$

Zamijenimo u jednačini:

y sa ky

y' sa ky'

y'' sa ky''

$\vdots$

Ako se nakon toga jed. podijeli sa k^2 i dobijemo istu jednačinu kao polaznu, jed. je homogena.

$$ky \cdot ky'' = (ky')^2 + 15(ky)^2 \sqrt{x}$$

$$k^2 y y'' = k^2 (y')^2 + 15 k^2 y^2 \sqrt{x} \quad /: k^2$$

$$y y'' = (y')^2 + 15 y^2 \sqrt{x}$$ $\to$ ista kao polazna $\Rightarrow$ homogena

smjena promjenljive:

$$y = e^{\int z(x)\,dx} \qquad z \text{ - nova f-ja}$$

$$\Rightarrow y' = \underbrace{e^{\int z(x)\,dx}}_{y} \cdot z(x)$$

$$y' = y \cdot z \Rightarrow y'' = y'z + y \cdot z'$$

$$= y \cdot z \cdot z + y \cdot z'$$

$$= y(z^2 + z')$$

$$y \cdot y(z^2 + z') = y^2 z^2 + 15 y^2 \sqrt{x} \quad /: y^2 \qquad y \neq 0$$

$$z^2 + z' = z^2 + 15\sqrt{x}$$

$$z' = 15\sqrt{x}$$

$$z = \int 15\sqrt{x}\,dx = 15 \cdot \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + A = \frac{2}{3}\cancel{15}^{5} \sqrt{x^3} + A = 10\sqrt{x^3} + A$$

$z = \frac{y'}{y}$

$\frac{y'}{y} = 10\sqrt{x^3} + A \quad / dx, \int$

$\int \frac{dy}{y} = \int (10x^{\frac{3}{2}} + A)\,dx$

$\ln|y| = 10 \cdot \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + Ax + \ln B$

$\ln y - \ln B = 4\sqrt{x^5} + Ax$

$\ln \frac{y}{B} = \ln e^{4\sqrt{x^5} + Ax}$

$y = B \cdot e^{4\sqrt{x^5} + Ax}$ opšte rj.

$B = 0 \Rightarrow y = 0 \rightarrow$ spada u opšte rj.

b) $y(xy'' + y') = x(y')^2(1-x)$

$\cancel{b}y(x \cdot by'' + by') = x(by')^2(1-x)$

$b^2 y(xy'' + y') = b^2 x(y')^2(1-x) \quad / : b^2$

$y(xy'' + y') = x(y')^2(1-x)$

$y = e^{\int z(x)\,dx} \Rightarrow y' = y \cdot z \Rightarrow y'' = y(z^2 + z')$

$y \cdot [xy(z^2 + z') + yz] = xy^2 z^2 (1-x)$

$y^2 [x(z^2 + z') + z] = xy^2 z^2 (1-x) \quad / : y^2 \quad y \neq 0$

$x(z^2 + z') + z = xz^2(1-x)$

$\cancel{xz^2} + xz' + z = \cancel{xz^2} - x^2 z^2$ možemo $/ : x \Rightarrow$ Bernulijeva ili:

$(xz)' = -(xz)^2$

$xz = u, \quad u = u(x) \rightarrow$ nova f-ja

$u' = -u^2 \quad / : (-u^2)$

$$\frac{u'}{-u^2} = 1 \quad \Big/ dx, \int$$

$$\int \frac{du}{-u^2} = \int dx$$

$$\frac{1}{u} = x + A \qquad \Big/ \, u = xz \, \Big/$$

$$\frac{1}{xz} = x + A$$

$$z = \frac{y'}{y} \Rightarrow \frac{1}{z} = \frac{y}{y'}$$

$$\frac{1}{x} \cdot \frac{y}{y'} = x + A$$

$$y' = \frac{y}{x(x+A)}$$

$$\frac{y'}{y} = \frac{1}{x(x+A)} \qquad \Big/ dx, \int$$

$$\int \frac{dy}{y} = \int \frac{dx}{x(x+A)}$$

$$\frac{1}{x(x+A)} = \frac{1}{A} \cdot \frac{x + A - x}{x(x+A)} = \frac{1}{A}\left(\frac{1}{x} - \frac{1}{x+A}\right)$$

$$\ln|y| = \frac{1}{A} \ln\left[\left|\frac{x}{x+A}\right| \cdot \underbrace{B}_{\ln B}\right] \qquad \Big/ \cdot A$$

$$\ln|y^A| = \ln\left|\frac{Bx}{x+A}\right|$$

$$y^A = \frac{Bx}{x+A} \rightarrow \text{opšte rj.}$$

Ako je $B = 0 \Rightarrow y = 0 \rightarrow$ nije sing. rj.

Za vježbu: probati i metodu mamjestanja izvoda

c) $x^2 y y'' + (y')^2 = 0$

d) $x^2 y y'' = (y - xy')^2$

e) $x y y'' = y'(y + y')$

f) $x^2 \left[(y')^2 - 2 y y'' \right] = y^2$

LINEARNE DIFERENCIJALNE JEDNAČINE VIŠEG REDA SA KONSTANTNIM KOEFICIJENTIMA

ove jed. mogu biti najmanje 2. reda

$$y^{(n)}+a_1y^{(n-1)}+a_2y^{(n-2)}+\dots+a_{n-1}y''+a_ny=f(x)\ \dots (*)$$

Ovo je nehomogena lin. D.J. n-tog reda sa konstantnim koeficijentima.

$a_1, a_2, \dots, a_n \rightarrow$ **koeficijenti**

$f(x) \rightarrow$ **slobodni član**

Ako je $f(x)\equiv 0$ onda je $(*)$ homogena.

rj. nehomogene zavisi od rješavanja homogene pa rješavamo homogenu:

$n=2$:

$$y''+a_1y'+a_2y=0 \rightarrow \text{polazna jed.}$$

rj. se obično piše u $e^{\lambda x}$ obliku:

$y=e^{\lambda x} \Rightarrow y'=\lambda e^{\lambda x}, \quad y''=\lambda^2 e^{\lambda x}$

$\lambda^2+a_1\lambda+a_2=0$ - karakteristična jednačina polazne jed.

1° Ako su $\lambda_1,\lambda_2\in\mathbb{R}$ i $\lambda_1\neq\lambda_2$ onda je opšte rj.:

$$y=C_1e^{\lambda_1x}+C_2e^{\lambda_2x}$$

2° $\lambda_1=\lambda_2\in\mathbb{R} \Rightarrow y=(C_1+C_2x)\,e^{\lambda_1x}$

3° $\lambda_1,\lambda_2\notin\mathbb{R} \Rightarrow \lambda_{1,2}=\alpha\pm\beta i \quad (\alpha,\beta\in\mathbb{R})$

$$y=e^{\alpha x}(C_1\cos\beta x+C_2\sin\beta x)$$

1. a) $y'' + y' - 2y = 0$

odgovarajuća karak. jed. je:

$\lambda^2 + \lambda - 2 = 0 \qquad \lambda_1 = 1 \quad \lambda_2 = -2$

$y = C_1 e^{x} + C_2 e^{-2x}$

b) $4y'' + 4y' + y = 0$

k. jed.: $4\lambda^2 + 4\lambda + 1 = 0$

$(2\lambda + 1)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = -\frac{1}{2}$

$y = C_1 e^{-\frac{1}{2}x} + C_2 x e^{-\frac{1}{2}x}$ ili:

$y = e^{-\frac{1}{2}x}(C_1 + C_2 x)$

c) $y'' - 2y' + 10y = 0$

k. jed.: $\lambda^2 - 2\lambda + 10 = 0 \qquad \lambda_{1,2} = \frac{2 \pm \sqrt{4-40}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$

$y = e^{x}(C_1 \cos 3x + C_2 \sin 3x)$

d) $y''' - 8y = 0$

k. jed. $\lambda^3 - 8 = 0 \qquad (\lambda^3 - 2^3) = 0 \qquad (\lambda - 2)(\lambda^2 + 2\lambda + 4) = 0$

/ ili $\lambda = \sqrt[3]{8} \rightarrow$ preko trigonometrije /

?

$\lambda_1 = 2 \qquad \lambda_{2,3} = \frac{-2 \pm \sqrt{4-16}}{2} = \frac{-2 \pm 2\sqrt{3}i}{2} = -1 \pm \sqrt{3}i$

$y = C_1 e^{2x} + e^{-x}(C_2 \cos\sqrt{3}x + C_3 \sin\sqrt{3}x)$

e) $y^{IV} + 2 \cdot y'' + y = 0$

kar. jed.: $r^4 + 2r^2 + 1 = 0$

$$(r^2+1)^2 = 0$$

$$r^2 = -1 \Rightarrow r = \pm i$$

$r_{1,2} = i \quad r_{3,4} = -i$ jer je $(r^2+1)^2 \Rightarrow$ dvostruka rj.

Kada je $r = \pm i \Rightarrow y = e^{0x}(C_1 \cos x + C_2 \sin x) = C_1 \cos x + C_2 \sin x$

bez ponavljanja rj.

Ukupno:

$$\underline{\underline{y = (C_1 + C_2 x)\cos x + (C_3 + C_4 x)\sin x}}$$

Za vježbu:

f) $2y'' - 5y' + 2y = 0$ $\qquad y = c_1 e^{\frac{1}{2}x} + c_2 e^{2x}$ $\qquad r_1 = \frac{1}{2} \quad r_2 = 2$

g) $y^{IV} - y = 0$ $\qquad y = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$ $\qquad r_1 = 1, r_2 = -1, r_{3,4} = \pm i$

h) $y^{V} - 6y^{IV} + 9y''' = 0$ $\qquad y = c_1 + c_2 x + c_3 x^2 + (c_4 + c_5 x) e^{3x}$ $\qquad r_{1,2,3} = 0, r_{4,5} = 3$

i) $y''' - y'' - y' + y = 0$ $\qquad y = c_1 e^{-x} + (c_2 + c_3) e^{x}$ $\qquad r_1 = -1 \quad r_{2,3} = 1$

j) $y^{IV} - 5y'' + 4y = 0$ $\qquad y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-2x}$ $\qquad r_{1,2} = \pm 1 \quad r_{3,4} = \pm 2$

k) $y''' - 3y' + 2y = 0$ $\qquad y = (c_1 + c_2 x) e^{x} + c_3 e^{-2x}$ $\qquad r_1 = r_2 = 1 \quad r_3 = -2$

* Za nehomogene D.J. višeg reda (linearne)

Opšte rješenje: $y = y_p + y_h$

$y_h \to$ opšte rj. odgovarajuće homogene jed.

$y_p \to$ neko partikularno rj. nehomogene jed.

2. Riješiti nehomogene lin. jed.:

a) $y'' - 2y' - 3y = e^{4x}$

$y'' - 2y' - 3y = 0$

k. jed. $\lambda^2 - 2\lambda - 3 = 0$

$\lambda_1 = 3, \ \lambda_2 = -1$

$y_h = c_1 e^{3x} + c_2 e^{-x}$

Sada tražimo y_p (kakvog je oblika desna strana tako uzimamo i za partikularno)

$y_p = a e^{4x} (\cdot x^s)$

obavezno provjeravamo šta stoji uz x tj. 4 i da li je $4 = \lambda_1$ ili λ_2

Ako je $4 = \lambda_1$ ili $4 = \lambda_2 \Rightarrow y_p = a e^{4x} \cdot x$

ako je $4 = \lambda_1 = \lambda_2 \Rightarrow y_p = a e^{4x} \cdot x^2$

s - višestrukost rj. k. jed. (u našem slučaju $s = 0$)

$y_p' = a \cdot e^{4x} \cdot 4 = 4ae^{4x}$

$y_p'' = 4ae^{4x} \cdot 4 = 16ae^{4x}$

$y_p'' - 2y_p' - 3y_p = e^{4x}$

$16ae^{4x} - 8ae^{4x} - 3ae^{4x} = e^{4x} \ / : e^{4x}$

$5a = 1 \Rightarrow a = \frac{1}{5}$

$y_p = \frac{1}{5} e^{4x}$

$y = \frac{1}{5} e^{4x} + c_1 e^{3x} + c_2 e^{-x}$

b) $y'' + y' - 2y = 3xe^x$

$y'' + y' - 2y = 0$

$\lambda^2 + \lambda - 2 = 0 \qquad \underline{\underline{\lambda_1 = 1}}, \lambda_2 = -2$

$y_h = C_1 e^x + C_2 e^{-2x}$

$3x \to$ polinom 1. reda $(ax+b)$

$y_p = (ax+b) \cdot e^x \cdot x$ zbog podudaranja broja uz x u e^x sa rješenjem k. jed.

$y_p = (ax^2 + bx)\,e^x$

$y_p' = (2ax + b)e^x + (ax^2 + bx)\,e^x$

$y_p' = (ax^2 + 2ax + bx + b)\,e^x$

$y_p'' = (2ax + 2a + b)\,e^x + (ax^2 + 2ax + bx + b)\,e^x$

$y_p'' = e^x(ax^2 + 4ax + bx + 2a + 2b)$

$y_p'' + y_p' - 2y_p = 3xe^x$

$(ax^2 + 4ax + bx + 2a + 2b)e^x + (ax^2 + 2ax + bx + b)e^x - 2(ax^2 + bx)e^x = 3xe^x \quad /:e^x$

$\cancel{2ax^2} + 6ax + \cancel{2bx} + 2a + 3b - \cancel{2ax^2} - \cancel{2bx} = 3x$

$6ax + 2a + 3b = 3x$

$6a = 3 \Rightarrow \boxed{a = \frac{1}{2}}$

$2a + 3b = 0 \Rightarrow \cancel{2} \cdot \frac{1}{\cancel{2}} + 3b = 0 \Rightarrow \boxed{b = -\frac{1}{3}}$

$y_p = \left(\frac{1}{2}x^2 - \frac{1}{3}x\right)e^x$

$y = \left(\frac{1}{2}x^2 - \frac{1}{3}x\right)e^x + C_1 e^x + C_2 e^{-2x}$

c) $y'' + y = x \sin x$

$y'' + y = 0$

k.j. $\lambda^2 + 1 = 0$

$\lambda^2 = -1$

$\lambda_{1,2} = \pm i$

$y_h = e^{0x}(C_1 \cos x + C_2 \sin x)$

$y_h = C_1 \cos x + C_2 \sin x$

$y_p = \left[(ax+b)\cos x + (cx+d)\sin x\right] \cdot x$

kada imamo ili sin ili cos ili oboje u y_p moramo uključiti oboje

$\sin x = e^{0x} \cdot \sin x \longmapsto 0 \pm i = \pm i$

ako nemamo $e^{\alpha x}$ možemo staviti e^{0x}

Napomena: Ako se na desnoj str. jed. pojavljuje izraz oblika $e^{\alpha x} \cos \beta x$ ili $e^{\alpha x} \sin \beta x$, desnoj strani pridružuje se konjugovano kompleksni par $\alpha \pm \beta i$ i poredi se sa rješenjima karakteristične jed.

$y_p = (ax^2 + bx)\cos x + (cx^2 + dx)\sin x$

$y_p' = (2ax + b)\cos x + (ax^2 + bx)(-\sin x) + (2cx + d)\sin x + (cx^2 + dx)\cos x$

$= \cos x\,(cx^2 + dx + 2ax + b) + \sin x\,(-ax^2 - bx + 2cx + d)$

$y_p'' = -\sin x\,(cx^2 + dx + 2ax + b) + \cos x\,(2cx + d + 2a) +$

$+ \cos x\,(-ax^2 - bx + 2cx + d) + \sin x\,(-2ax - b + 2c)$

$= \cos x\,(-ax^2 + bx + 4cx + 2a + 2d) + \sin x\,(-cx^2 - 4ax - dx -$

$- 2b + 2c)$

Sada uvrstimo ovo u polaznu jed.:

$y_p'' + y_p = x\sin x$

$\cos x\left(\cancel{-ax^2} - \cancel{bx} + 4cx + 2a + 2d\right) + \sin x\left(\cancel{-cx^2} - 4ax - \cancel{dx} - 2b + 2c\right) +$

$+ \cos x\left(\cancel{ax^2} + \cancel{bx}\right) + \sin x\left(\cancel{cx^2} + \cancel{dx}\right) = x\sin x$

$\cos x(4cx + 2a + 2d) + \sin x(-4ax - 2b + 2c) = x\sin x$

$\cos x$ i $\sin x$ su lin. nezavisne pa posmatramo:

$$\left.\begin{array}{l} \text{uz } \cos x: \ 4cx + 2a + 2d = 0 \\ \text{uz } \sin x: \ -4ax - 2b + 2c = x \end{array}\right\} \Rightarrow \begin{cases} 4c = 0 \\ 2a + 2d = 0 \\ -4a = 1 \\ -2b + 2c = 0 \end{cases}$$

$c = 0, \quad a = -\frac{1}{4}, \quad b = 0, \quad d = -a = \frac{1}{4}$

$y_p = -\frac{1}{4}x^2 \cdot \cos x + \frac{1}{4}x\sin x \quad \Rightarrow \quad y = y_p + y_H$

$y = \frac{1}{4}\left(-x^2\cos x + x\sin x\right) + C_1\cos x + C_2\sin x$

$y = \cos x\left(C_1 - \frac{1}{4}x^2\right) + \sin x\left(C_2 + \frac{1}{4}x\right)$

d) $y'' - 5y' = 3x^2 + \sin 5x$

$y'' - 5y' = 0 \overset{k.j.}{\Rightarrow} \lambda^2 - 5\lambda = 0$

$\lambda(\lambda - 5) = 0 \Rightarrow \lambda_1 = 0 \quad \lambda_2 = 5$

$y_h = C_1 e^{0x} + C_2 e^{5x}$

$y_h = C_1 + C_2 e^{5x}$

Ukoliko na desnoj str. imamo zbir više sabiraka različitog tipa, imamo polinom 2. stepena $3x^2$ i trig. f-ju $\sin 5x$ to dvoje ko nebo i zemlja :-).
Pa imamo part. rj. koliko imamo sabiraka ovih - tj. 2.:

$y_p = y_{p_1} + y_{p_2}$

$y_{p_1} = (ax^2+bx+c)\cdot x$ podmolara se to rj sa 0 uz x

$3x^2 = 3x^3 e^{0x}$ uz x stoji 0 pa je 0 jedno rj. k. jed.

$y_{p_1} = ax^3 + bx^2 + cx$

$y'_{p_1} = 3ax^2 + 2bx + c$

$y''_{p_1} = 6ax + 2b$

$y''_{p_1} - 5y'_{p_1} = 3x^2$

$6ax + 2b - 5(3ax^2 + 2bx + c) = 3x^2$

$6ax + 2b - 15ax^2 - 10bx - 5c = 3x^2$

$$\Rightarrow \begin{cases} -15a = 3 \\ 6a - 10b = 0 \\ 2b - 5c = 0 \end{cases}$$

$a = -\frac{3}{15} = -\frac{1}{5}$

$6a = 10b \qquad b = \frac{6a}{10} = \frac{3a}{5}$

$b = -\frac{\frac{3}{5}}{5} = -\frac{3}{25}$

$2b = 5c \Rightarrow c = \frac{2b}{5} = -\frac{6}{125}$

$y_{p_1} = -\frac{1}{5}x^3 - \frac{3}{25}x^2 - \frac{6}{125}x$

Imamo realna rj. k. jed. i sin x - ne poredimo to (kao mi slučaj imag. rj. i bez sin x ili cos x):

$$\boxed{y_{p_2} = d\sin 5x + f\cos 5x}$$

$$y'_{p_2} = 5d\cos 5x - 5f\sin 5x$$

$$y''_{p_2} = -25d\sin 5x - 25f\cos 5x$$

$$y''_{p_2} - 5y'_{p_2} = \sin 5x$$

$$-25d\sin 5x - 25f\cos 5x - 5(5d\cos 5x - 5f\sin 5x) = \sin 5x$$

$$\sin 5x(-25d + 25f) + \cos x(-25f - 25d) = \sin 5x$$

$$\left.\begin{aligned} -25d + 25f &= 1 \\ -25f - 25d &= 0 \end{aligned}\right\} +$$

$$-50d = 1$$

$$d = -\frac{1}{50}$$

$$-25(d+f) = 0 \Rightarrow f = -d = \frac{1}{50}$$

$$y_{p_2} = -\frac{1}{50}\sin 5x + \frac{1}{50}\cos 5x$$

$$y = C_1 + C_2 e^{5x} - \frac{1}{5}x^3 - \frac{3}{25}x^2 - \frac{6}{125}x + \frac{1}{50}(\cos 5x - \sin 5x)$$

za vježbu:

e) $y'' + 2y' - 3y = x^2 e^x$ $\qquad y = C_1 e^x + C_2 e^{-3x} + \left(\frac{1}{12}x^3 - \frac{1}{16}x^2 + \frac{1}{32}x\right)e^x$

f) $y'' - 4y' + 8y = e^{2x} + \sin 2x$ $\qquad y = e^{2x}(C_1\cos 2x + C_2\sin 2x) + \frac{1}{4}e^{2x} + \frac{9}{79}\sin 2x + \frac{4}{79}\cos 2x$

g) $y'' + y = 4\sin x$

h) $y'' + y = 2e^x - x^2$

i) $y'' + 2y' + 2y = xe^{-x}$, $y(0) = y'(0) = 0$ $\qquad y = e^{-x}(\sin x - x)$

j) $y''' - 3y' - 2y = 9e^{2x}$, $y(0) = 0$, $y'(0) = -3$, $y''(0) = 3$

k) $y^{IV} + y'' = 2\cos x$, $y(0) = -2$, $y'(0) = 1$, $y''(0) = y'''(0) = 0$

uslovi - vezani za Vronskijan ?

2. Odrediti oblik partikularnog rj.

a) $y'' + 6y' + 10y = 3xe^{-3x} - 2e^{3x}\cos x$

$y'' + 6y' + 10y = 0$

k. j. $r^2 + 6r + 10 = 0$

$r_{1,2} = \frac{-6 \pm \sqrt{36-40}}{2} = \frac{-6 \pm 2i}{2}$

$r_{1,2} = -3 \pm 1i$

$y_h = e^{-3x}(c_1 \cos x + c_2 \sin x)$

$y_p = y_{p_1} + y_{p_2}$

e^{-3x} → pol. 1. stepena

$y_{p_1} = (ax+b)e^{-3x}$

r_1 i r_2 su konjugovano kompl. rj. a to nije vezano za e^{-3x} pa ne dodajemo x nista više na y_{p_1}

$e^{3x}\cos x \longmapsto 3 \pm i$

$y_{p_2} = e^{3x}(c \cdot \cos x + d \sin x)$

$y_p = (ax+b)e^{-3x} + e^{3x}(c\cos x + d\sin x)$

$-2\, e^{3x} \cos x$

je predviđeno u rj. a i da je nema o bio bi isti oblik

b) $y'' - 8y' + 20y = 5x e^{4x} \sin 2x$

u y_p polinom i trig. fja idu zajedno

$y'' - 8y' + 20y = 0$

k.j. $\lambda^2 - 8\lambda + 20 = 0$

$$\lambda_{1,2} = \frac{8 \pm \sqrt{64 - 80}}{2} = \frac{8 \pm 4i}{2} = 4 \pm 2i$$

$$y_h = e^{4x}(c_1 \cos 2x + c_2 \sin 2x)$$

$$y_p = e^{4x}\left[(ax+b)\sin 2x + (cx+d)\cos 2x\right] \cdot x$$

$$e^{4x} \sin 2x \longmapsto 4 \pm 2i$$

$$y_p = e^{4x}\left[(ax^2+bx)\sin 2x + (cx^2+dx)\cos 2x\right]$$

c) $y'' - 2y' + y = 2x e^{x} + e^{x} \sin 2x$

$y'' - 2y' + y = 0$

$\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0 \qquad \lambda_1 = \lambda_2 = 1$

$$y_h = (c_1 + c_2 x)e^x$$

$$y_p = y_{p_1} + y_{p_2}$$

$\lambda_1 = \lambda_2 = 1$

$$y_{p_1} = (ax+b)e^{1x} \cdot x^2$$

$$y_{p_1} = (ax^3 + bx^2)e^x$$

$$y_{p_2} = e^x(c \cos 2x + d \sin 2x)$$

d) $y'' - 4y' + 5y = e^{2x} \sin^2 x$

$y'' - 4y' + 5y = 0$

$r^2 - 4r + 5 = 0$

$r_{1,2} = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$

$y_H := e^{2x}(c_1 \cos x + c_2 \sin x)$

$e^{2x} \sin^2 x = e^{2x} \cdot \frac{1-\cos 2x}{2} = \frac{1}{2}\left(e^{2x} - e^{2x}\cos 2x\right) =$

$= \frac{1}{2} e^{2x} - \frac{1}{2} e^{2x} \cos 2x$

$y_P = y_{P_1} + y_{P_2}$

$y_{P_1} = a e^{2x}$

$y_{P_2} = e^{2x}(b \cos 2x + c \sin 2x)$

$e^{2x} \cos 2x \xrightarrow{\text{njemu odgovara}} 2 \pm 2i$ a mi imamo $2 \pm i$

pa ne treba ništa više dodavati

e) $y^{IV} + 5y'' + 4y = \sin x \cdot \cos 2x$

$y^{IV} + 5y'' + 4y = 0$

$r^4 + 5r^2 + 4 = 0$

$r^2 = t \Rightarrow t^2 + 5t + 4 = 0 \qquad t_{1,2} = \frac{-5 \pm \sqrt{25-16}}{2} = \frac{-5 \pm 3}{2} \begin{cases} -4 \\ -1 \end{cases}$

$r^2 = -4 \Rightarrow r_{1,2} = \pm 2i$

$r^2 = -1 \Rightarrow r_{3,4} = \pm i$

$y_h = C_1 \cos 2x + C_2 \sin 2x + C_3 \cos x + C_4 \sin x$

$$\sin x \cdot \cos 2x = \frac{1}{2}\left[\sin(x+2x) + \sin(x-2x)\right]$$

$$= \frac{1}{2}(\sin 3x - \sin x)$$

$$= \frac{1}{2}\sin 3x - \frac{1}{2}\sin x$$

$\pm 3i$ $\quad$ $\pm i$

$\notin$ rješenjima

$y_p = y_{p_1} + y_{p_2}$

$y_{p_1} = a \sin 3x + b \cos 3x$

$y_{p_2} = (c \sin x + d \cos x) \cdot x$

za vježbu:

f) $y'' + 7y' + 10y = xe^{-2x} \cdot \cos 5x$ $\qquad y_p = e^{-2x}\left[(ax+b)\cos 5x + (cx+d)\sin 5x\right]$

g) $y'' - 2y' + 5y = 2xe^{x} + e^{x} \sin 2x$

h) $y'' - 8y' + 17y = e^{4x}(x^2 - 3x \sin x)$

i) $y''' + y' = \sin x + x\cos x$

j) $y'' + 3y' + 2y = e^{-x} \cdot \cos^2 x$

Metoda varijacije konstanti

$y'' + a_1 y' + a_2 y = f(x)$

Postavi se homogena: $y'' + a_1 y' + a_2 y = 0$

$$y_h = c_1 y_1(x) + c_2 y_2(x)$$

Ako postavimo fju oblika:

(*) $y = C_1(x)\, y_1(x) + C_2(x)\, y_2(x) \to$ opće rj. polazne jed.

$$\left.\begin{aligned} c_1' y_1 + c_2' y_2 &= 0 \\ c_1' y_1' + c_2' y_2' &= f(x) \end{aligned}\right\}$$ ovo je obični sus. lin. jed. po nepoznatim c_1' i c_2'.

Njegovim rješavanjem dobijemo

$c_1' = \alpha(x) \Rightarrow C_1 = \int \alpha(x)\,dx$

$c_2' = \beta(x) \Rightarrow C_2 = \int \beta(x)\,dx$

Onda c_1 i c_2 uvrstimo u (*)

1. $y'' - 2y' + y = \dfrac{e^x}{x}$

$y'' - 2y' + y = 0$

K.j.: $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0 \quad r_1 = r_2 = 1$

$$y_h = c_1 \cdot \underbrace{e^x}_{y_1} + c_2 \underbrace{x e^x}_{y_2}$$

$c_1' e^x + c_2' \cdot x e^x = 0 \quad / : e^x$

$c_1' e^x + c_2' (e^x + x e^x) = \dfrac{e^x}{x} \quad / : e^x$

$$-\begin{cases} c_1' + c_2' \cdot x = 0 \\ c_1' + c_2' (1+x) = \dfrac{1}{x} \end{cases}$$

$$C_2'(1+x) - C_2' x = \frac{1}{x} \qquad C_1' + \frac{1}{x}\cdot x = 0$$

$$C_2'(1+x-x) = \frac{1}{x} \qquad C_1' = -1$$

$$C_2' = \frac{1}{x}$$

$$C_2(x) = \int \frac{dx}{x} = \ln|x| + B \qquad C_1(x) = \int -dx = -x + A$$

$$y = C_1(x)\cdot e^x + C_2(x)\, x e^x = (A-x)e^x + (B + \ln|x|)\, x e^x =$$

$$= e^x(\underbrace{A}_{konst} - x + x\ln|x| + \underbrace{B}_{konst.} x) \longrightarrow \text{opće rj.}$$

2. $y''' + y' = \operatorname{tg} x$

$y''' + y' = 0$

k.j. $r^3 + r = 0$

$r(r^2+1) = 0 \qquad r_1 = 0 \qquad r^2 = -1 \Rightarrow r_{2,3} = \pm i$

$$e^{0x}$$

$$y_h = C_1 \cdot 1 + C_2 \cos x + C_3 \sin x$$

Opće rj. polazne jed. glasi:

$$y = C_1(x) + C_2(x)\cos x + C_3(x)\sin x$$

$$y_1 = 1, \quad y_2 = \cos x, \quad y_3 = \sin x, \quad f(x) = \operatorname{tg} x$$

$$\begin{cases} C_1' y_1 + C_2' y_2 + C_3' y_3 = 0 \\ C_1' y_1' + C_2' y_2' + C_3' y_3' = 0 \\ C_1' y_1'' + C_2' y_2'' + C_3' y_3'' = f(x) \end{cases}$$

$$\left.\begin{aligned} c_1' + c_2'\cos x + c_3'\sin x &= 0 \\ -c_2'\sin x + c_3'\cos x &= 0 \\ -c_2'\cos x - c_3'\sin x &= \operatorname{tg} x \end{aligned}\right\} (*)$$

za sis. (*):

$$D = \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} = \sin^2 x + \cos^2 x = 1 \Rightarrow \text{sigurno imamo 1 rješenje}$$

$$D_1 = \begin{vmatrix} 0 & \cos x \\ \operatorname{tg} x & -\sin x \end{vmatrix} = -\cos x \cdot \operatorname{tg} x = -\sin x$$

$$D_2 = \begin{vmatrix} -\sin x & 0 \\ -\cos x & \operatorname{tg} x \end{vmatrix} = -\sin x \cdot \operatorname{tg} x = \frac{-\sin^2 x}{\cos x}$$

$$c_2' = \frac{D_1}{D} = D_1 = -\sin x$$

$$c_3' = \frac{D_2}{D} = D_2 = \frac{-\sin^2 x}{\cos x}$$

$$c_1' = -c_2'\cos x - c_3'\sin x$$

$$c_1' = \sin x\cos x + \frac{\sin^3 x}{\cos x} = \frac{\sin x\cos^2 x + \sin^3 x}{\cos x} = \frac{\sin x(\cos^2 x + \sin^2 x)}{\cos x}$$

$$c_1' = \operatorname{tg} x$$

$$c_1 = \int \operatorname{tg} x\, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx = -\ln|\cos x| + A$$

$$c_2 = \int -\sin x\, dx = \cos x + B$$

$$c_3 = \int \frac{-\sin^2 x}{\cos x} dx = -\int \frac{1-\cos^2 x}{\cos x} dx = -\int \frac{dx}{\cos x} + \int \cos x\, dx =$$

Prema univerzalnoj smjeni:

$$\Big/ \sin\alpha = \cos\beta : \quad \alpha+\beta = 90^\circ \quad \beta = 90^\circ - x \Big/$$

$$\int \frac{1}{\sin x}dx = \ln\left|tg\frac{x}{2}\right| + C$$

$$\int \frac{1}{\cos x}dx = \int \frac{dx}{\sin\left(\frac{\pi}{2}-x\right)} = \left| \begin{matrix} \frac{\pi}{2}-x = t \\ dx = -dt \end{matrix} \right| = -\int \frac{dt}{\sin t} = -\ln\left|tg\frac{t}{2}\right| + C$$

$$= -\ln\left|tg\left(\frac{\frac{\pi}{2}-x}{2}\right)\right| + C = -\ln\left|tg\left(\frac{\pi}{4}-\frac{x}{2}\right)\right| + C =$$

$$= \ln\left|tg\left(\frac{\pi}{4}-\frac{x}{2}\right)\right|^{-1} + C = \ln\left|ctg\left(\frac{\pi}{4}-\frac{x}{2}\right)\right| + C =$$

$$= \ln\left|tg\left(\frac{\pi}{2}-\left(\frac{\pi}{4}-\frac{x}{2}\right)\right)\right| + C = \ln\left|tg\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + C$$

$$C_2 = \sin x - \ln\left|tg\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| - C$$

$$y = -\ln|\cos x| + A + (B + \cos x)\cos x + \left(\sin x - \ln\left|tg\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + C\right)\sin x$$

za vježbu:

3. $y'' + 3y' + 2y = \frac{1}{e^x+1}$ $\qquad y = \left(\ln|e^x+1| + A\right)e^{-x} + \left(\ln|e^x+1| - e^x + B\right)e^{-2x}$

4. $y'' + 4y = ctg\,x$ $\qquad y = \left(\frac{1}{2}\sin 2x - x + A\right)\cos 2x + \sin 2x\left[\frac{1}{4}\left(\ln|\cos x| - \frac{5}{4}\cos 2x\right) + B\right]$

5. $y'' + 2y' + y = 3e^{-x}\sqrt{x+1}$

6. $x^3(y' - y) = x^2 - 2$

uraditi

Napomena: $y' - y = \frac{x^2-2}{x^3} \Rightarrow \Big| \, (*) = \frac{x^2-2}{x^3}$

LINEARNE D.J. VIŠEG REDA SA PROMJENLJIVIM KOEFICIJENTIMA

(1) $y^{(n)} \cdot a_n(x) + y^{(n-1)} a_{n-1}(x) + \dots + y' a_1(x) + y \cdot a_0(x) = f(x)$

Ako $f(x) \neq 0$ jed. (1) se zove **nehomogena**.
Ako je $f(x) = 0$ jed. (1) se zove **homogena** lin. D.J. višeg reda sa promjenljivim koeficijentima.

* Rješavanje homogene jed.:

Neka je $y_1 = y_1(x)$ jedno **partikularno** rj. te jed.

Uzimamo **smjenu**: $y = y_1 \cdot z \Rightarrow y' = y_1' \cdot z + y_1 \cdot z'$, $y'' = \dots$, $y''' = $
sa ovom smjenom jed. se svodi za 1 stepen
Ovom smjenom se snižava red jed. za 1.

1. $x^2(x+1)y'' - 2y = 0$

$$y_1 = 1 + \frac{1}{x}$$

$$y = y_1 \cdot z$$

$$y' = y_1' z + y_1 \cdot z'$$

$$y' = -\frac{1}{x^2} \cdot z + \left(1 + \frac{1}{x}\right) \cdot z'$$

$$y'' = y_1' z' + y_1'' \cdot z + y_1' \cdot z' + y_1 \cdot z''$$

$$y'' = y_1'' \cdot z + 2y_1' z' + y_1 \cdot z''$$

$$y'' = \frac{2}{x^3} \cdot z + 2 \cdot \left(-\frac{1}{x^2}\right) z' + \left(1 + \frac{1}{x}\right) z''$$

$$x^2(x+1)\left[\frac{2}{x^3} \cdot z - \frac{2}{x^2} z' + \frac{x+1}{x} \cdot z''\right] - 2 \cdot \left(1 + \frac{1}{x}\right) \cdot z = 0$$

$$\cancel{\frac{2}{x}(x+1)\cdot z} - 2(x+1)z' + x(x+1)^2 z'' - \cancel{2\cdot\frac{x+1}{x}\cdot z} = 0$$

uvijek se izgube članovi sa z.

$$x(x+1)^2 z'' = 2(x+1)z'$$

Ako je $x+1=0$ ova jed. je zadovoljena ali polazna jed. nije (nije identitet)

$x+1=0 \Rightarrow$ polazna jed. nije zadovoljena

$x+1 \neq 0 \Rightarrow \quad x(x+1)z'' = 2z' \quad / : z' x(x+1)$

$$\frac{z''}{z'} = \frac{2}{x(x+1)}$$

$$\frac{(z')'}{z'} = \frac{2}{x(x+1)} \quad / dx, \int$$

$$\int \frac{dz'}{z'} = \int \frac{2}{x(x+1)}$$

$$\ln|z'| = 2\int\left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$\frac{1}{x} - \frac{1}{x+1} = \frac{\cancel{x}+1-\cancel{x}}{x(x+1)} = \frac{1}{x(x+1)}$$

$$\ln|z'| = 2\cdot(\ln|x| - \ln|x+1|) + \ln A$$

$$\ln|z'| = \ln A\cdot\frac{x^2}{(x+1)^2}$$

$$z' = \frac{Ax^2}{(x+1)^2}$$

$$z = A\cdot\int\frac{x^2}{(x+1)^2}dx$$

$$\int \frac{x^2}{(x+1)^2}\,dx = \int \frac{(x+1)^2-(2x+1)}{(x+1)^2}\,dx = \int 1 - \frac{2x+2-1}{(x+1)^2}\,dx =$$

$$= x - \int \frac{2(x+1)}{(x+1)^2}\,dx + \int \frac{1}{(x+1)^2}\,dx =$$

$$= x - 2\ln|x+1| - \frac{1}{x+1} + C$$

$$z = A\cdot\left(x - 2\ln|x+1| - \frac{1}{x+1}\right) + B \qquad \cancel{B = A\cdot C}$$

$$y = y_1\cdot z \Rightarrow z = \frac{y}{y_1} = \frac{y}{1+\frac{1}{x}}$$

$$z = \frac{y}{\frac{x+1}{x}} = y\cdot\frac{x}{x+1}$$

$$y = z\cdot\frac{x+1}{x}$$

$$y = A\cdot\frac{x+1}{x}\left(x - 2\ln|x+1| - \frac{1}{x+1}\right) + B\cdot\frac{x+1}{x}$$

$$y = A(x+1) - 2A\,\frac{(x+1)\ln|x+1|}{x} - \frac{A}{x} + B\cdot\frac{x+1}{x}$$

2. $(2x+1)y'' + 4x\cdot y' - 4y = 0$

$y_1 = e^{\alpha x}$ za neko $\alpha\in\mathbb{R}$

$$y_1' = e^{\alpha x}\cdot\alpha$$
$$y_1'' = e^{\alpha x}\cdot\alpha^2$$

$$(2x+1)\alpha^2 e^{\alpha x} + 4x\,\alpha e^{\alpha x} - 4e^{\alpha x} = 0 \quad /:e^{\alpha x}$$

$$(2x+1)\alpha^2 + 4x\alpha - 4 = 0$$

$$x(2\alpha^2 + 4\alpha) + \alpha^2 - 4 = 0$$

$$\Rightarrow \begin{cases} 2\alpha^2+4\alpha=0 \\ \alpha^2-4=0 \end{cases}$$

$$2\alpha(\alpha+2)=0$$

$$\alpha^2=4$$

$$\alpha_1=0 \quad \alpha_2=-2$$

$$\alpha_{1,2}=\pm 2$$

$\Rightarrow \boxed{\alpha=-2}$ – zajedničko

$$\boxed{y_1=e^{-2x}}$$

$$y_1'=-2e^{-2x}$$

$$y_1''=4e^{-2x}$$

$$y=y_1\cdot z$$

$$y'=y_1'z+y_1\cdot z'$$

$$y'=-2e^{-2x}z+e^{-2x}\cdot z'=e^{-2x}(-2z+z')$$

$$y''=y_1\cdot z''+2y_1'z'+y_1''\cdot z$$

$$y''=e^{-2x}\cdot z''-4\cdot e^{-2x}\cdot z'+4\cdot e^{-2x}\cdot z=e^{-2x}(z''-4z'+4z)$$

$$(2x+1)\,e^{-2x}(z''-4z'+4z)+4x\cdot e^{-2x}(-2z+z')-4\cdot e^{-2x}\cdot z=0 \;/:e^{-2x}$$

$$(2x+1)z''-4(2x+1)\cdot z'+\cancel{4(2x+1)z}-\cancel{8xz}+4xz'-\cancel{4z}=0$$

$$(2x+1)z''=(-4x+8x+4)\,z'$$

$$(2x+1)z''=(4x+4)\,z'$$

$$\frac{z''}{z'}=\frac{4x+4}{2x+1} \quad /dx,\int$$

$$\int\frac{dz'}{z'}=\int\frac{4x+4}{2x+1}\,dx$$

$$\int \frac{4x+4}{2x+1}\,dx = \int \frac{4x+2+2}{2x+1}\,dx = \int \frac{4x+2}{2x+1}\,dx + \int \frac{2}{2x+1}\,dx =$$

$$= 2x + \ln|2x+1| + \ln A$$

$$\ln|z'| = \underbrace{2x}_{\ln e^{2x}} + \ln A\cdot|2x+1|$$

$$\ln|z'| = \ln Ae^{2x}|2x+1|$$

$$z' = Ae^{2x}(2x+1)$$

$$z = A\int e^{2x}(2x+1)\,dx = \left| \begin{array}{ll} u = 2x+1 & dv = e^{2x}dx \\ du = 2dx & v = \frac{1}{2}e^{2x} \end{array} \right| =$$

$$= A\left[(2x+1)\cdot\frac{1}{2}e^{2x} - \int e^{2x}dx\right] =$$

$$= A\left[\frac{2x+1}{2}e^{2x} - \frac{1}{2}e^{2x}\right] + B$$

$$z = A\,\frac{\cancel{2}xe^{2x} + \cancel{e^{2x}} - \cancel{e^{2x}}}{\cancel{2}} + B = Axe^{2x} + B$$

$$y = e^{-2x}\cdot z = Ax + Be^{-2x}$$

za vježbu:

3. $xy'' + 2y' - xy = 0$

$y_1 = \frac{e^x}{x}$ $\qquad y = \frac{e^x}{x}\cdot e^{(A-2)x}\cdot B$

4. $y'' - 2y(1+tg^2 x) = 0$

$y_1 = tg\,x$ $\qquad y = -1 - x\,tg\,x + Ax\,tg\,x + B\,tg\,x$

5. $x(x^2+6)y'' - 4(x^2+3)y' + 6xy = 0$ $\qquad$ naći izvode i uvrstiti

y_1 je polinom 2. stepena $\qquad y_1 = ax^2 + bx + c$

6. $xy''' - y'' - xy' + y = 0$

$y_1 = x, \quad y_2 = e^x$

iskoristimo 1. rj. i svedemo na 2. red, pa 2. rj. $\Rightarrow$ 1. reda

EULEROVA JEDNAČINA

$$\sum_{k=0}^{m} a_k x^k y^{(k)} = f(x)$$

sin, cos, polinom, e^x (+, −)

ali ne može :, stepenovanje korijen

$y^{(0)} = y$

$$a_0 y + a_1 x y' + a_2 x^2 y'' + \dots + a_m x^m y^{(m)} = f(x) \quad \dots (1)$$

smjena: $x = e^t$,

$y = x^r, \quad r$ - const.

(r - neodređena konstanta koju treba odrediti)

$y = (e^t)^r = e^{rt}$ uvrstimo $x = e^t$

$y' = r x^{r-1} = r e^{t(r-1)}$ $\quad y' = \frac{dy}{dx}$

~~$y' = \frac{dy}{dt} = r e^{t(r-1)}$~~

$y'' = r(r-1) x^{r-2} = r(r-1) e^{t(r-2)}$

$y''' = r(r-1)(r-2) x^{r-3} = r(r-1)(r-2) e^{t(r-3)}$

1. rješavamo homogenu:

$(1) \Rightarrow a_0 e^{rt} + a_1 e^t r e^{t(r-1)} + a_2 e^{2t} \cdot r(r-1) e^{t(r-2)} + \dots = 0 \quad /: e^{rt}$

$a_0 + a_1 r + a_2 r(r-1) + \dots = 0 \quad \dots (2)$ - karakteristična jed. jednačine (1)

1. $x^2y'' - 4xy' + 6y = 0$

$$e^{2t} \cdot r(r-1)e^{t(r-2)} - 4 \cdot e^{t} \cdot re^{t(r-1)} + 6e^{rt} = 0 \quad / : e^{rt}$$

$$r(r-1) - 4 \cdot r + 6 = 0$$

$$r^2 - 5r + 6 = 0$$

$$r_{1,2} = \frac{5 \pm \sqrt{25-24}}{2} = \frac{5 \pm 1}{2} \begin{cases} 2 \\ 3 \end{cases}$$

$r_1 = 2 \qquad r_2 = 3$

$$\dot{y} = \frac{dy}{dt}, \quad \ddot{y} = \frac{d^2y}{dt^2}, \ldots$$

Na osnovu polazne => kar. jed. => $y(t)$:

$\ddot{y} - 5\dot{y} + 6y = 0$ (za f-ju $y = y(t)$)

$y = c_1 e^{2t} + c_2 e^{3t}$ – opće rj. jed. $y(t)$

$e^{2t} = (e^t)^2 = x^2$
$e^{3t} = (e^t)^3 = x^3$ } vraćamo se na x

$\Rightarrow y = c_1 x^2 + c_2 x^3$ opće rj. početne jed.

2. $x^2y'' - xy' + y = 8x^3$

$x^2y'' - xy' + y = 0$ – odgovarajuća homogena

k.j.: $r(r-1) - r + 1 = 0$

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0$$

$$r_1 = r_2 = 1$$

$$y_h = (c_1 + c_2 t)e^t$$

opće rj.: $y = y_p + y_h$

$\ddot{y} - 2\dot{y} + y = 8e^{3t}$ (iz k.j. lijeva str., i desna $8x^3 = 8e^{3t}$)

$y_p = ae^{3t}$

$\dot{y}_p = 3ae^{3t}$

$\ddot{y}_p = 9ae^{3t}$

$9ae^{3t} - 6ae^{3t} + ae^{3t} = 8e^{3t}$

$4ae^{3t} = 8e^{3t} \quad / : 4e^{3t}$

$a = 2$

$y_p = 2e^{3t}$

$y = 2e^{3t} + (c_1 + c_2 t)e^{t}$ $\quad / \; e^{t} = x \Rightarrow t = \ln x \; /$

$y = 2x^3 + (c_1 + c_2 \ln x) \cdot x$

3. $x^2 y'' - 6y = 5x^3 + 8x^2$

$x^2 y'' - 6y = 0$

$\lambda(\lambda - 1) - 6 = 0$

$\lambda^2 - \lambda - 6 = 0$

$\lambda_{1,2} = \frac{1 \pm \sqrt{1+24}}{2} = \frac{1 \pm 5}{2} \begin{cases} -2 \\ 3 \end{cases}$

$y_h = c_1 e^{-2t} + c_2 e^{3t}$

$\ddot{y} - \dot{y} - 6y = 5e^{3t} + 8e^{2t}$

$y_p = ae^{3t} + be^{2t}$

$\dot{y}_p = 3ae^{3t} + 2be^{2t}$

$\ddot{y}_p = 9ae^{3t} + 4be^{2t}$

$9ae^{3t} + 4be^{2t} - 3ae^{3t} + 2be^{2t} - 6ae^{3t} - 6be^{2t} = 5e^{3t} + 8e^{2t}$

$-4be^{2t} = 5e^{3t} + 8e^{2t} \quad / : e^{2t}$

$-4b = 5e^{t} + 8$

$y_p = y_{p_1} + y_{p_2}$

$y_{p_1} = ae^{3t} \cdot t$

$y_{p_2} = be^{2t}$

$y'_{p_1} = ae^{3t}(3t+1)$

$y''_{p_1} = 3ae^{3t}(3t+1) + ae^{3t} \cdot 3 = 3ae^{3t}(3t+2)$

$3ae^{3t}(3t+2) - ae^{3t}(3t+1) - 6ae^{3t} \cdot t = 5e^{3t} \quad / : e^{3t}$

$\cancel{9at} + 6a - \cancel{3at} - a - \cancel{6at} = 5$

$5a = 5$

$a = 1$

$y_{p_1} = t \cdot e^{3t}$

$y'_{p_2} = 2be^{2t}$

$y''_{p_2} = 4be^{2t}$

$4be^{2t} - 2be^{2t} - 6be^{2t} = 8e^{2t} \quad / : e^{2t}$

$4b - 2b - 6b = 8$

$-4b = 8$

$b = -2$

$y_{p_2} = -2e^{2t}$

$y_p = te^{3t} - 2e^{2t}$

$y = te^{3t} - 2e^{2t} + C_1e^{2t} + C_2e^{3t} \Rightarrow y = \ln x \cdot x^3 - 2x^2 + C_1x^{-2} + C_2x^3$

4. $(2x+3)^3 \cdot y''' + 3(2x+3)y' - 6y = 0$

$$\sum_{k=0}^{m} a_k (ax+b)^k y^{(k)} = 0$$

$2x+3 = e^t, \quad y = (2x+3)^r = e^{rt}$

$y' = r(2x+3)^{r-1} \cdot 2 = 2r(e^t)^{r-1} = 2re^{t(r-1)}$

$y'' = 2r(r-1)(2x+3)^{r-2} \cdot 2 = 4r(r-1)e^{t(r-2)}$

$y''' = 4r(r-1)(r-2)(2x+3)^{r-3} \cdot 2 = 8r(r-1)(r-2)e^{t(r-3)}$

$\cancel{e^{3t}} \cdot 8r(r-1)(r-2)e^{t(r-3)} + 3\cancel{e^t} \cdot 2re^{t(r-1)} - 6e^{tr} = 0 \quad / : e^{tr}$

k.j.: $\quad 8r(r-1)(r-2) + 6r - 6 = 0$

$8r(r-1)(r-2) + 6(r-1) = 0$

$2(r-1)\left[4r(r-2)+3\right] = 0$

$r_1 = 1; \quad 4r^2 - 8r + 3 = 0$

$D = 64 - 48 = 16$

$r_{2,3} = \frac{8 \pm 4}{8} = \begin{cases} \frac{3}{2} \\ \frac{1}{2} \end{cases} \qquad r_2 = \frac{3}{2} \qquad r_3 = \frac{1}{2}$

$y = c_1 e^t + c_2 e^{\frac{3}{2}t} + c_3 e^{\frac{1}{2}t}$

$y = c_1 e^t + c_2 \sqrt{e^{3t}} + c_3 \sqrt{e^t}$

$y = c_1(2x+3) + c_2\sqrt{(2x+3)^3} + c_3\sqrt{2x+3}$

$y = \sqrt{2x+3}\left[c_1\sqrt{2x+3} + c_2(2x+3) + c_3\right]$

5. $x^2 y'' - 2y = \frac{3x^2}{x+1}$

nećemo moći koristiti metod partikularnih rj. jer imamo racionalnu fju pa možemo koristiti metod varijacije konst.

$x^2 y'' - 2y = 0$

k. j. $r(r-1) - 2 = 0$

$r^2 - r - 2 = 0 \Rightarrow r_1 = 2, \ r_2 = -1$

$y_h = c_1 e^{2t} + c_2 e^{-t}$

$\ddot{y} - \dot{y} - 2y = \frac{3e^{2t}}{e^t + 1}$

Opće rj. polazne jed. tražimo u obliku:

$y = c_1(t) \underbrace{e^{2t}}_{y_1} + c_2(t) \underbrace{e^{-t}}_{y_2}$

$c_1' \cdot y_1 + c_2' \cdot y_2 = 0 \ ; \qquad c_1' \cdot y_1' + c_2' \cdot y_2' = f(x)$

$$\left.\begin{aligned} c_1' \cdot e^{2t} + c_2' \cdot e^{-t} \\ c_1' \cdot 2e^{2t} + c_2' \cdot (-e^{-t}) = \frac{3e^{2t}}{e^t + 1} \end{aligned}\right\} +$$

$3 c_1' e^{2t} = \frac{3e^{2t}}{e^t + 1} \quad / : 3e^{2t}$

$c_1' = \frac{1}{e^t + 1}$

$c_2' e^{-t} = -c_1' e^{2t} \quad / \cdot e^t$

$c_2' = -c_1' e^{3t}$

$c_2' = -\frac{e^{3t}}{e^t + 1}$

$$c_1(t) = \int \frac{1}{e^t+1}\,dt \underset{1\cdot e^{-t}}{\overset{1\cdot e^{-t}}{=}} \int \frac{e^{-t}}{1+e^{-t}}\,dt = -\int \frac{-e^{-t}}{1+e^{-t}}\,dt = -\ln(1+e^{-t}) + A$$

$$c_2(t) = \int \frac{-e^{3t}}{e^t+1}\,dt = -\int \frac{e^t \cdot e^{2t}}{e^t+1}\,dt = \left| \begin{array}{l} e^t+1 = z \\ e^t dt = dz \\ e^t = z-1 \end{array} \right| = -\int \frac{(z-1)^2}{z}\,dz =$$

$$= -\int \frac{z^2-2z+1}{z}\,dz = -\frac{z^2}{2} + 2z - \ln|z| + B = \Big/ z = e^t+1 \Big/$$

$$= -\frac{(e^t+1)^2}{2} + 2(e^t+1) - \ln(e^t+1) + B =$$

$$= \frac{-e^{2t} - 2e^t - 1 + 4e^t + 4}{2} - \ln(e^t+1) + B$$

$$= \frac{2e^t - e^{2t} + 3}{2} - \ln(e^t+1) + B$$

$$y = \left[A - \ln(1+e^{-t})\right] e^{2t} + \left[\frac{2e^t - e^{2t} + 3}{2} - \ln(e^t+1) + B\right] e^{-t}$$

$$y = Ae^{2t} - e^{2t}\ln(1+e^{-t}) + \frac{2+3e^{-t}-e^{t}}{2} - e^{-t}\ln(1+e^t) + Be^{-t}$$

$$\Big/ x = e^t \Big/$$

$$y = Ax^2 - x^2\ln\left(1+\frac{1}{x}\right) + 1 + \frac{3}{2x} - \frac{x}{2} - \frac{\ln(1+x)}{x} + \frac{B}{x}$$

za vježbu:

6. $x^3y''' + xy' - y = 0$ $\qquad y = (c_1 + c_2\ln x + c_3(\ln x)^2)\,x$

7. $x^2y'' - 2y = \sin(\ln x)$ $\qquad y = c_1x^{-1} + c_2x^2 - \frac{3}{10}\sin\ln x + \frac{1}{10}\cos\ln x$

8. $(x-2)^2y'' - 3(x-2)y' + 4y = x$ $\qquad y = (c_1 + c_2\ln(x-2))(x-2)^2 + \frac{1}{2} + (x-2)$

9. $y' - \frac{2y}{x^2} = 3\ln x$ $\qquad$ } ne valja $\qquad y = \frac{c_1}{x} + c_2x^2 - \frac{1}{2}x\ln x$

10. $x^2y'' - xy' + y = \frac{\ln x}{x} + \frac{x}{\ln x}$

SISTEMI LINEARNIH D.J. SA KONSTANTNIM KOEFICIJENTIMA

$x_1 = x_1(t), \quad x_2 = x_2(t), \ldots, x_n = x_n(t)$

$x_i \to$ nepoznate f-je

$t \to$ argument (nezavisna promjenljiva)

Prije rješavanja, nastoji se svesti na sljedeći oblik:

$$(*) \begin{cases} \dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + f_1(t) \\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + f_2(t) \\ \vdots \\ \dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + f_n(t) \end{cases}$$

$a_{ij} \in \mathbb{R}$; a_{ij} - koeficijenti sistema

$f_i(t)$ - slobodni članovi

$f_i(t) \equiv 0 \Rightarrow (*)$ je homogeni sis.

Za rješavanje ovih sis. koristimo dvije metode:

1° metoda svođenja na D.J. sa jednom nepoznatom f-jom

2° Ojlerova metoda (rješavanje pomoću karakteristične jed.

4) Metoda svođenja na DJ sa jednom nepoznatom fjom

sistem (*) za $n=2$:

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + a_{12}x_2 + f_1(t) & \dots (1) \\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2 + f_2(t) & \dots (2) \end{cases}$$

Iz (1) $\Rightarrow$ $x_2 = \dfrac{\dot{x}_1 - a_{11}x_1 - f_1(t)}{a_{12}} \quad \dots (3) \Big/ \dfrac{d}{dt}$

$$\Rightarrow \dot{x}_2 = \frac{\ddot{x}_1 - a_{11}\dot{x}_1 - f_1'(t)}{a_{12}}$$

Ako uvrstimo zadnju i jednakost (3) u (2), dobijemo D.J. po nepoznatoj x_1 (linearna D.J. 2. reda sa konstantnim koef.); riješimo po x_1 i uvrstimo u (3).

Alternativno: iz (2) $\Rightarrow x_1 \Rightarrow \dot{x}_1$ pa uvrstimo u (1)

$\Rightarrow x_2 \Rightarrow \dot{x}_2$

1. $\dot{x} = 2x + y + 2e^{t} \quad \dots (4)$

   $\dot{y} = x + 2y - 3e^{4t} \quad \dots (5)$

Iz (4) $\Rightarrow$ $y = \dot{x} - 2x - 2e^{t}$ (6) $\Rightarrow$ $\dot{y} = \ddot{x} - 2\dot{x} - 2e^{t}$

$\overset{(5)}{\Rightarrow}$ $\ddot{x} - 2\dot{x} - 2e^{t} = x + 2\dot{x} - 4x - 4e^{t} - 3e^{4t}$

$$\ddot{x} - 4\dot{x} + 3x = -2e^{t} - 3e^{4t}$$

$$\ddot{x} - 4\dot{x} + 3x = 0$$

k.j. $\lambda^2 - 4\lambda + 3 = 0$

$$\lambda_1 = 1, \quad \lambda_2 = 3$$

$$x_h = c_1 e^{t} + c_2 e^{3t}$$

$x_p = x_{p_1} + x_{p_2}$

$x_{p_1} = ae^t \cdot t$ uz t stoji 1 a $\lambda = 1$ pa ide $\cdot t^1$

$\dot{x}_{p_1} = a(e^t \cdot t + e^t) = ae^t(t+1)$

$\ddot{x}_{p_1} = a[e^t(t+1) + e^t] = ae^t(t+2)$

$\ddot{x}_{p_1} - 4\dot{x}_{p_1} + 3x_{p_1} = -2e^t$

$ae^t(t+2) - 4ae^t(t+1) + 3ae^t \cdot t = -2e^t \ / : e^t$

$a(t+2) - 4a(t+1) + 3at = -2$

$a(t+2-4t-4+3t) = -2$

$-2a = -2 \Rightarrow a = 1$

$x_{p_1} = te^t$

$x_{p_2} = be^{4t} \Rightarrow \dot{x}_{p_2} = 4be^{4t}, \quad \ddot{x}_{p_2} = 16be^{4t}$

$\ddot{x}_{p_2} - 4\dot{x}_{p_2} + 3x_{p_2} = -3e^{4t}$

$\cancel{16be^{4t}} - \cancel{16be^{4t}} + 3be^{4t} = -3e^{4t} \ / : 3e^{4t}$

$b = -1 \Rightarrow x_{p_2} = -e^{4t}$

$x_p = te^t - e^{4t} \Rightarrow x = te^t - e^{4t} + C_1e^t + C_2e^{3t}$

$x = (t + C_1)e^t - e^{4t} + C_2e^{3t}$

Iz (6) $\Rightarrow y = e^t + (t+C_1)e^t - 4e^{4t} + 3C_2e^{3t} - 2(t+C_1)e^t + 2e^{4t} - 2C_2e^{3t} - 2e^t$

$y = e^t(1 + t + C_1 - 2t - 2C_1 - 2) + C_2e^{3t} - 2e^{4t}$

$y = (-t - C_1 - 1)e^t + C_2e^{3t} - 2e^{4t}$

Rješenje:

$$\begin{cases} x = (t + C_1)e^t - e^{4t} + C_2e^{3t} \\ y = -(t + C_1 + 1)e^t + C_2e^{3t} - 2e^{4t} \end{cases}$$

Za $n=3$:

$$\dot{x} = a_{11}x + a_{12}y + a_{13}z + f_1(t) \quad \ldots (1)$$
$$\dot{y} = a_{21}x + a_{22}y + a_{23}z + f_2(t) \quad \ldots (2)$$
$$\dot{z} = a_{31}x + a_{32}y + a_{33}z + f_3(t) \quad \ldots (3)$$

$$\ddot{x} = a_{11}\dot{x} + a_{12}\dot{y} + a_{13}\dot{z} + f_1'(t) \overset{(1),(2),(3)}{=}$$

uvrstimo (1), (2), (3) i grupišemo šta stoji uz x, y i z

$$= b_1x + b_2y + b_3z + g(t)$$

$$\dddot{x} = b_1\dot{x} + b_2\dot{y} + b_3\dot{z} + g'(t) \overset{(1),(2),(3)}{=} c_1x + c_2y + c_3z + h(t)$$

$$(*)\begin{cases} \dot{x} = a_{11}x + a_{12}y + a_{13}z + f_1(t) & \ldots (1) \\ \ddot{x} = b_1x + b_2y + b_3z + g(t) & \ldots (4) \\ \dddot{x} = c_1x + c_2y + c_3z + h(t) & \ldots (5) \end{cases}$$

Ako odaberemo dvije jed. između jed. (1), (4) i (5) i riješimo taj sis. po nepoznatim f-jama y i z, dobit ćemo y i z izražene preko x i izvoda od x. To se onda uvrsti u preostalu jednačinu sis. $(*)$ i odatle dobijemo jed. po nepoznatoj x.

2. $\dot{x} = 3x - y + z \quad \ldots (1)$

$\dot{y} = -x + 5y - z \quad \ldots (2)$

$\dot{z} = x - y + 3z \quad \ldots (3)$

$$\ddot{x} = 3\dot{x} - \dot{y} + \dot{z} = 3\cdot(3x - y + z) - (-x + 5y - z) + (x - y + 3z) =$$

$$= 9x - 3y + 3z + x - 5y + z + x - y + 3z =$$

$$= 11x - 9y + 7z$$

$$\dddot{x} = 11\dot{x} - 9\dot{y} + 7\dot{z} = 33x - 11y + 11z + 9x - 45y + 9z +$$

$$+ 7x - 7y + 21z = 49x - 63y + 41z$$

postavljamo novi sis. ekvivalentan polaznom:

$$\begin{cases} \dot{x} = 3x - y + z & \ldots (1) \\ \ddot{x} = 11x - 9y + 7z & \ldots (4) \\ \dddot{x} = 49x - 63y + 41z & \ldots (5) \end{cases}$$

iz (1) $\Rightarrow -y + z = \dot{x} - 3x \quad /\cdot(-7)$

iz (4) $\Rightarrow -9y + 7z = \ddot{x} - 11x$

$$-2y + 0 = -7\dot{x} + 10x + \ddot{x}$$

$$\boxed{y = -\frac{\ddot{x} - 7\dot{x} + 10x}{2}}$$

$$z = \dot{x} - 3x + y$$

$$\Rightarrow z = \dot{x} - 3x - \frac{\ddot{x} - 7\dot{x} + 10x}{2} = \frac{2\dot{x} - 6x - \ddot{x} + 7\dot{x} - 10x}{2}$$

$$\boxed{z = \frac{-\ddot{x} + 9\dot{x} - 16x}{2}}$$

iz (5) $\Rightarrow \dddot{x} = 49x - 63\,\frac{-\ddot{x} + 7\dot{x} - 10x}{2} + 41\cdot\frac{-\ddot{x} + 9\dot{x} - 16x}{2} \quad /\cdot 2$

$$2\dddot{x} = 98x + 63\ddot{x} - 441\dot{x} + 630x - 41\ddot{x} + 369\dot{x} - 656x$$

$$2\dddot{x} - 22\ddot{x} + 72\dot{x} - 72x = 0 \quad /:2$$

$\dddot{x} - 11\ddot{x} + 36\dot{x} - 36x = 0$ pa rješavamo po x:

k.j. $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

riješićemo preko Hornerove šeme:

$\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36\}$ → djelioci od 36

	1	−11	36	−36
	1	−2	−18	36
2	1	−9	18	0

	1	−11	36	−36
2	1	−9	18	0

$2\cdot(-9) = -18 + 36$

$2\cdot 18 = 36 - 36 = 0$

$\lambda_1 = 2$

$\lambda^2 - 9\lambda + 18 = 0 \Rightarrow \lambda_2 = 3,\ \lambda_3 = 6$

$x = c_1 e^{2t} + c_2 e^{3t} + c_3 e^{6t}$

$y = \dfrac{-\ddot{x} + 7\dot{x} - 10x}{2} = \ldots = c_2 e^{3t} - 2c_3 e^{6t}$

$z = \dfrac{-\ddot{x} + 9\dot{x} - 16x}{2} = \ldots = -c_1 e^{2t} + c_2 e^{3t} + c_3 e^{6t}$

3. $\dot{x} = x - 2y - z \quad \ldots (1)$

$\dot{y} = y - x + z \quad \ldots (2)$

$\dot{z} = x - z \quad \ldots (3)$

iz (3) $\Rightarrow$ $x = z + \dot{z} \Rightarrow \dot{x} = \dot{z} + \ddot{z}$

iz (1) $\Rightarrow$ $\cancel{\dot{z}} + \ddot{z} = \cancel{z} + \cancel{\dot{z}} - 2y - \cancel{z}$

$\ddot{z} = -2y \Rightarrow y = -\frac{1}{2}\ddot{z} \Rightarrow \dot{y} = -\frac{1}{2}\dddot{z}$

iz (2) $\Rightarrow$ $-\frac{1}{2}\dddot{z} = -\frac{1}{2}\ddot{z} - \cancel{z} - \dot{z} + \cancel{z} \quad / \cdot 2$

$-\dddot{z} = -\ddot{z} - 2\dot{z}$

$-\dddot{z} + \ddot{z} + 2\dot{z} = 0$

k.j. $-\lambda^3 + \lambda^2 + 2\lambda = 0$

$-\lambda(\lambda^2 - \lambda + 2) = 0 \quad \lambda_1 = 0,\ \lambda_2 = 2,\ \lambda_3 = -1$

$z = C_1 + C_2 e^{2t} + C_3 e^{-t}$

$\dot{z} = 2C_2 e^{2t} - C_3 e^{-t}$

$\ddot{z} = 4C_2 e^{2t} + C_3 e^{-t}$

uvrstimo $\dot{z} \Rightarrow x$ i y tj. x, y, z

$$\Rightarrow \begin{cases} x = C_1 + 3C_2 e^{2t} \\ y = -2C_2 e^{2t} - \frac{1}{2} C_3 e^{-t} \\ z = C_1 + C_2 e^{2t} + C_3 e^{-t} \end{cases}$$

za vježbu:

4\. a) $\dot{x} = 2x + y$

$\dot{y} = 3x + 4y$

b) $\dot{x} - 5x + 3y = 2e^{3t}$

$\dot{y} - x - y = 5e^{-t}$

c) $\frac{dy}{dx} = 3y - \frac{3}{2} z + e^{x}$ $\quad x$ je argument (kao t)

$\frac{dz}{dx} = 4y - 2z - e^{2x}$

d) $\dot{x} = 2x + y$

$\dot{y} = x + 3y - z$

$\dot{z} = 2y + 3z - x$

e) $\dot{x} = 2x - y - z$

$\dot{y} = 3x - 2y - 3z$

$\dot{z} = 2z - x + y$

f) $\dot{x} = y - x - 2z$ $\quad$ problem

$\dot{y} = x - 2y + 2z$

$\dot{z} = 3x - 3y + 5z$

2) Ojlerova metoda (Eulerova)

Rješavamo homogene sis.

$$\dot{x} = a_{11}x + a_{12}y$$
$$\dot{y} = a_{21}x + a_{22}y$$

Napomena: Kada radimo ovom metodom sis. mora biti **homogen**.

$x = Ae^{\lambda t}$, $y = Be^{\lambda t}$

A, B, λ - nepoznate konstante

$\dot{x} = A\lambda e^{\lambda t}$, $\dot{y} = B\lambda e^{\lambda t}$

$$A\lambda e^{\lambda t} = a_{11}Ae^{\lambda t} + a_{12}Be^{\lambda t} \quad / : e^{\lambda t}$$
$$B\lambda e^{\lambda t} = a_{21}Ae^{\lambda t} + a_{22}Be^{\lambda t} \quad / : e^{\lambda t}$$

$$A\lambda = a_{11}A + a_{12}\cdot B$$
$$B\lambda = a_{21}A + a_{22}\cdot B$$

$\Rightarrow \begin{array}{l} (\lambda - a_{11})\cdot A - a_{12}\cdot B = 0 \\ -a_{21}A + (\lambda - a_{22})\cdot B = 0 \end{array}$ } hom. lin. sis. sa nepoz. A i B i njeb ima rj. pa i trivijalno (0,0)

$D \neq 0 \Rightarrow (0,0)$

$D = 0 \Rightarrow \infty$ rj.

Pošto tražimo netrivijalna rj. hom. sis. lin. jednačina postavljamo uslov da je $D = 0$

$$\begin{vmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{vmatrix} = 0$$

$(\lambda - a_{11})(\lambda - a_{22}) - a_{21}\cdot a_{12} = 0 \rightarrow$ karakteristična jed. polaznog sistema

Imamo 3 slučaja:

1. λ_1, λ_2 realni i $\lambda_1 \neq \lambda_2$
2. kompleksni

1. $\dot{x} = x - y$

$\dot{y} = y - 4x$

$x = Ae^{\lambda t} \quad y = Be^{\lambda t}$

$\dot{x} = A\lambda$; $x = A$
$\dot{y} = B\lambda$; $y = B$ } zamijenimo

$A\lambda = A - B$

$B\lambda = B - 4A$

$A(\lambda - 1) + B = 0$

$4A + B(\lambda - 1) = 0$

$$\begin{vmatrix} \lambda-1 & 1 \\ 4 & \lambda-1 \end{vmatrix} = (\lambda-1)^2 - 4 = 0 \to \text{k. jed.}$$

$(\lambda-1)^2 = 4 \quad \lambda - 1 = \pm 2 \Rightarrow \lambda = 1 \pm 2 \Rightarrow \lambda_1 = 3 \quad \lambda_2 = -1$

Kada je $\lambda = 3$ sis. glasi: $2A + B = 0$

$4A + 2B = 0 \ /:2 \Rightarrow 2A + B = 0$

Ovo se uvijek treba desiti tj. u sis. od 2 jed. uvijek je jedna lin. kombinacija 2. - kada tražimo netrivijalna rj. ($D = 0$); a u slučaju sis. od 3 jed. imamo najmanje 1 jed. a nekad i dvije koje su lin. kom. druge 2, odnosno 1.

$B = -2A$

proizvoljno
$A = 1 \Rightarrow B = -2$

$x_1 = e^{3t}$

$y_1 = -2e^{3t}$

$\lambda_2 = -1 \Rightarrow -2A + B = 0$

$$4A - 2B = 0$$

$$B = 2A$$

$$A = 1 \Rightarrow B = 2$$

$$x_2 = e^{-t}$$

$$y_2 = 2e^{-t}$$

$$\boxed{\begin{aligned} x &= C_1 x_1 + C_2 x_2 \\ y &= C_1 y_1 + C_2 y_2 \end{aligned}}$$

$$x = C_1 e^{3t} + C_2 e^{-t}$$

$$y = -2C_1 e^{3t} + 2C_2 e^{-t}$$

2. $\dot{x} = x + z - y$

$\dot{y} = x + y - z$

$\dot{z} = 2x - y$

$x = Ae^{\lambda t}, \quad y = Be^{\lambda t}, \quad z = Ce^{\lambda t}$

Nakon $/ : e^{\lambda t}$ imamo:

$$A\lambda = A + C - B$$

$$B\lambda = A + B - C$$

$$C\lambda = 2A - B$$

$$A(\lambda - 1) + B - C = 0$$

$$-A + B(\lambda - 1) + C = 0$$

$$-2A + B + C\lambda = 0$$

$$\begin{vmatrix} \lambda-1 & 1 & -1 \\ -1 & \lambda-1 & 1 \\ -2 & 1 & \lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} \lambda-2 & 1 & -1 \\ 0 & \lambda-1 & 1 \\ \lambda-2 & 1 & \lambda \end{vmatrix} = (\lambda-2)\begin{vmatrix} 1 & 1 & -1 \\ 0 & \lambda-1 & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

$$(\lambda-2)\begin{vmatrix} 1 & 1 & -1 \\ 0 & \lambda-1 & 1 \\ 0 & 0 & \lambda+1 \end{vmatrix} = 0$$

$$(\lambda-2)(\lambda-1)(\lambda+1) = 0$$

$\underline{\lambda_1 = 2}$ $\underline{\lambda_2 = 1}$ $\underline{\lambda_3 = -1}$

Za $\forall \lambda_i$ $i=\overline{1,3}$ tražimo A, B i C:

Ako je $\lambda_1 = 2$: sis. glasi:
$$\left.\begin{aligned} A+B-C&=0 \\ -A+B+C&=0 \end{aligned}\right\}+$$
$$-2A+B+2C=0$$

$$2B=0 \quad B=0$$

$$A-C=0 \Rightarrow A=C=1$$

proizvoljno uzimamo $=1$

$$\begin{aligned} x_1 &= e^{2t} \\ y_1 &= 0 \\ z_1 &= e^{2t} \end{aligned}$$

još za $\lambda_2 = 1$, $\lambda_3 = -1$

$\lambda_2 = 1$ $\quad B - C = 0 \Rightarrow B = C$

$-A + C = 0 \Rightarrow A = C$

$-2A + B + C = 0$

$A = B = C = 1$

$x_2 = y_2 = z_2 = e^{t}$

$\lambda_3 = -1$

$$\left.\begin{aligned} -2A + B - C &= 0 \\ -A - 2B + C &= 0 \end{aligned}\right\} +$$

1. i 3. jed. su iste $\Rightarrow$ sis. ima ∞ rj.

$-2A + B - C = 0$

$-3A - B = 0$

$-3A = B$

$-2A - 3A - C = 0$

$-5A = C$

$A = 1 \Rightarrow B = -3, \quad C = -5$

$x_3 = e^{-t} \qquad y_3 = -3e^{-t} \qquad z_3 = -5e^{-t}$

Rješenje (konačno) sis. glasi:

$$x = C_1 e^{2t} + C_2 e^{t} + C_3 e^{-t}$$

$$y = C_2 e^{t} - 3C_3 e^{-t}$$

$$z = C_1 e^{2t} + C_2 e^{t} - 5C_3 e^{-t}$$

3. $\dot{x} = y + z$

$\dot{y} = x + z$

$\dot{z} = x + y$

$A\lambda = B + C$

$B\lambda = A + C$

$C\lambda = A + B$

$A\lambda - B - C = 0$

$-A + B\lambda - C = 0$

$-A - B + C\lambda = 0$

k. jed.
$$\begin{vmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix} = 0$$

ako je isti zbir po kolonama saberemo:

$\overline{I}_k + \overline{II}_k + \overline{III}_k$
$$\begin{vmatrix} \lambda-2 & -1 & -1 \\ \lambda-2 & \lambda & -1 \\ \lambda-2 & -1 & \lambda \end{vmatrix} = 0$$

$$(\lambda-2)\begin{vmatrix} 1 & -1 & -1 \\ 1 & \lambda & -1 \\ 1 & -1 & \lambda \end{vmatrix} = 0$$

$\overline{II}_v - \overline{I}_v$

$\overline{III}_v - \overline{I}_v$:
$$(\lambda-2)\begin{vmatrix} 1 & 1 & 1 \\ 0 & \lambda+1 & 0 \\ 0 & 0 & \lambda+1 \end{vmatrix} = 0$$

$$(\lambda-2)(\lambda+1)^2 = 0$$

$$\boxed{\lambda_1 = 2 \qquad \lambda_2 = \lambda_3 = -1}$$

jedno rj. se ponavlja imamo višestrukost 2; trebamo dobiti 2 trojke lin. nezavisnih rj.

$\lambda_1 = 2$

$$\left.\begin{aligned} 2A - B - C &= 0 \\ -A + 2B - C &= 0 \end{aligned}\right\} -$$

$$-A - B + 2C = 0$$

$$3A - 3B = 0 \quad / : 3$$

$$A - B = 0 \Rightarrow \underline{A = B}$$

$$2A - A - C = 0 \Rightarrow \underline{A = C}$$

$A = B = C = 1$

$x_1 = e^{2t}, \quad y_1 = e^{2t}, \quad z_1 = e^{2t}$

$\lambda = -1$

$$-A - B - C = 0 \quad / \cdot (-1)$$

$$-A - B - C = 0$$

$$-A - B - C = 0$$

$$A + B + C = 0$$

I $A = 1, \; B = 0, \; C = -1$ obično uzmemo da je jedna $= 0$

II $A = 0, \; B = 1, \; C = -1$

$x_2 = e^{-t}, \quad y_2 = 0, \quad z_2 = e^{-t}$

$x_3 = 0, \quad y_3 = e^{-t}, \quad z_3 = e^{-t}$

$$\Rightarrow \quad x = C_1 e^{2t} + C_2 e^{-t}$$

$$y = C_1 e^{2t} + C_3 e^{-t}$$

$$z = C_1 e^{2t} + (C_2 + C_3) e^{-t}$$

2. način:

za $\lambda = -1$

$$x = (A_1 + A_2 t) e^{-t}$$
$$y = (B_1 + B_2 t) e^{-t}$$
$$z = (D_1 + D_2 t) e^{-t}$$

pri čemu su $A_1, A_2, B_1, B_2, D_1, D_2$ neodređene konst. koje treba izraziti preko C_2 i C_3;
ovo treba uvrstiti u polazni sis.:

$$\dot{x} = y + z$$
$$\dot{y} = x + z$$
$$\dot{z} = x + y$$

$$\dot{x} = A_2 e^{-t} + (A_1 + A_2 t) e^{-t} (-1)$$
$$\dot{x} = (A_2 - A_1 - A_2 t) e^{-t}$$
$$\dot{y} = (B_2 - B_1 - B_2 t) e^{-t}$$
$$\dot{z} = (D_2 - D_1 - D_2 t) e^{-t}$$

$$(A_2 - A_1 - A_2 t) e^{-t} = (B_1 + B_2 t) e^{-t} + (D_1 + D_2 t) e^{-t} \quad / : e^{-t}$$
$$(B_2 - B_1 - B_2 t) e^{-t} = (A_1 + A_2 t) e^{-t} + (D_1 + D_2 t) e^{-t} \quad / : e^{-t}$$
$$(D_2 - D_1 - D_2 t) e^{-t} = (A_1 + A_2 t) e^{-t} + (B_1 + B_2 t) e^{-t} \quad / : e^{-t}$$

$$\left.\begin{aligned} A_2 - A_1 - A_2 t &= B_1 + D_1 + (B_2 + D_2) t \\ B_2 - B_1 - B_2 t &= A_1 + D_1 + (A_2 + D_2) t \\ D_2 - D_1 - D_2 t &= A_1 + B_1 + (A_2 + B_2) t \end{aligned}\right\} \text{jednakost polinoma}$$

$$A_2 - A_1 = B_1 + D_1 \qquad -A_2 = B_2 + D_2$$
$$B_2 - B_1 = A_1 + D_1 \qquad -B_2 = A_2 + D_2$$
$$D_2 - D_1 = A_1 + B_1 \qquad -D_2 = A_2 + B_2$$

$$\left.\begin{aligned} A_2 &= A_1 + B_1 + D_1 \\ B_2 &= A_1 + B_1 + D_1 \\ D_2 &= A_1 + B_1 + D_1 \end{aligned}\right\} \Rightarrow A_2 = B_2 = D_2$$

$$\begin{aligned} 0 &= A_2 + B_2 + D_2 \\ 0 &= A_2 + B_2 + D_2 \\ 0 &= A_2 + B_2 + D_2 \end{aligned}$$

$0 = A_2 + A_2 + A_2 \Rightarrow 3A_2 = 0 \Rightarrow \underline{A_2 = 0}$

$\underline{B_2 = D_2 = 0}$

$\Rightarrow A_1 + B_1 + D_1 = 0$ jer su $A_2 = B_2 = D_2 = 0$

$A_1 = c_2,\ B_1 = c_3 \Rightarrow D_1 = -A_1 - B_1 = \underbrace{-c_2 - c_3}$

Rješenje:

$$x = c_1 e^{2t} + c_2 e^{-t}$$

$$y = c_1 e^{2t} + c_3 e^{-t}$$

$$z = \underbrace{c_1 e^{2t}}_{z_1} - \underbrace{(c_2 + c_3) e^{-t}}_{z_{2,3}}$$

Ako imamo $r_1 = r_2 = r_3 = 3$

$$x = (A_1 + A_2 t + A_3 t^2) e^{3t}$$

$$y = (B_1 + B_2 t + B_3 t^2) e^{3t}$$

$$z = (D_1 + D_2 t + D_3 t^2) e^{3t}$$

i imamo onda 3 konst. za odrediti,

izrazimo ih preko c_1, c_2, c_3

4. $\dot{x} + 5x - 3y = 0$

$\dot{y} + 15x - 7y = 0$

$x = Ae^{\lambda t}, \quad y = Be^{\lambda t}$

$A\lambda + 5A - 3B = 0$

$B\lambda + 15A - 7B = 0$

$A(\lambda+5) - 3B = 0$

$15A + B(\lambda-7) = 0$

k. jed. $\begin{vmatrix} \lambda+5 & -3 \\ 15 & \lambda-7 \end{vmatrix} = 0$

$(\lambda+5)(\lambda-7) + 45 = 0$

$\lambda^2 + 5\lambda - 7\lambda - 35 + 45 = 0$

$\lambda^2 - 2\lambda + 10 = 0$

$\lambda_{1,2} = \frac{2 \pm \sqrt{4-40}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$

$\lambda = 1 + 3i$: $A(6+3i) - 3B = 0 \quad /:3$

$15A + B(-6+3i) = 0 \quad /:3$

$A(2+i) - B = 0$

$5A + B(-2+i) = 0$

$B = A(2+i)$

$A = 1 \Rightarrow B = 2+i$

privremena rj.:

$\overline{x_1} = e^{(1+3i)t} = e^t \cdot e^{3it}$

$\overline{y_1} = (2+i)e^{(1+3i)t} = (2+i)e^t \cdot e^{3it}$

$\lambda = 1 - 3i$:
$$A(-3i+6) - 3B = 0 \quad /:3$$
$$15A + B(-6-3i) = 0 \rightarrow \text{posljedica } 1$$
$$A(2-i) - B = 0$$
$$B = A(2-i)$$
$$A = 1 \Rightarrow B = 2-i$$

$$\overline{x}_2 = e^{(1-3i)t}$$
$$\overline{y}_2 = (2-i)\, e^{(1-3i)t}$$

$e^{\pm ix} = \cos x \pm i\sin x \rightarrow$ Ojlerov oblik

$$\overline{x}_1 + \overline{x}_2 = e^t(\cos 3t + \cancel{i\sin 3t}) + e^t(\cos 3t - \cancel{i\sin 3t})$$
$$= 2e^t \cdot \cos 3t$$
$$\overline{x}_1 - \overline{x}_2 = 2ie^t \sin 3t$$

$$x_1 = \frac{\overline{x}_1 + \overline{x}_2}{2} = e^t \cos 3t$$

$$x_2 = \frac{\overline{x}_1 - \overline{x}_2}{2i} = e^t \sin 3t$$

$$\overline{y}_1 + \overline{y}_2 = (2+i)\, e^t(\cos 3t + i\sin 3t) + (2-i)\, e^t(\cos 3t - i\sin 3t)$$
$$= e^t\left[\cos 3t\,(2+\cancel{i}+2-\cancel{i}) + i\sin 3t\,(\cancel{2}+i-\cancel{2}+i)\right]$$
$$= e^t(4\cos 3t - 2\sin 3t) = 2e^t(2\cos 3t - \sin 3t)$$
$$\overline{y}_1 - \overline{y}_2 = e^t\left[\cos 3t\,(\cancel{2}+i-\cancel{2}+i) + i\sin 3t\,(2+\cancel{i}+2-\cancel{i})\right]$$
$$= e^t(2i\cos 3t + 4i\sin 3t) = 2ie^t(\cos 3t + 2\sin 3t)$$

$$y_1 = \frac{\overline{y}_1 + \overline{y}_2}{2} = e^t(2\cos 3t - \sin 3t)$$

$$y_2 = \frac{\overline{y}_1 - \overline{y}_2}{2i} = e^t(\cos 3t + 2\sin 3t)$$

Rješenje:

$$x = c_1 e^t \cos 3t + c_2 e^t \sin 3t$$
$$y = c_1 e^t (2\cos 3t - \sin 3t) + c_2 e^t (\cos 3t + 2\sin 3t)$$

$$x = e^t (c_1 \cos 3t + c_2 \sin 3t)$$
$$y = e^t [\cos 3t (2c_1 + c_2) + \sin 3t (-c_1 + 2c_2)]$$

Za vježbu:

a) $\dot{x} + \cancel{x} - 8y = 0$

$\dot{y} - x - y = 0$

b) $\dot{x} = x - 3y$

$\dot{y} = 3x + y$

c) $\dot{x} = 2y - 3x$

$\dot{y} = y - 2x$

d) $\dot{x} = 2x - y + z$

$\dot{y} = x + 2y - z$

$\dot{z} = x - y + 2z$

Rj: $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3$

e) $\dot{x} = 4x - y - z$

$\dot{y} = x + 2y - z$

$\dot{z} = x - y + 2z$

$\lambda_1 = 2,\ \lambda_{2,3} = 3$

f) $\dot{x} = 2x + y$

$\dot{y} = x + 3y - z$

$\dot{z} = 2y + 3z - x$

$\lambda_1 = 2, \ \lambda_{2,3} = 3 \pm i$

$\boxed{g)}$ $\dot{x} = 2x + 2z - y$

$\dot{y} = x + 2z$ uraditi

$\dot{z} = y - 2x - z$

$\lambda_1 = 1, \ \lambda_{2,3} = \pm i$

RJEŠAVANJE NEHOMOGENIH SISTEMA METODOM VARIJACIJE KONSTANTI i Ojlerovom metodom (za hom. sis.)

1. $\dot{x} - 4x - y = -3e^t$

$\dot{y} + 2x - y = -2e^t$

$\dot{x} - 4x - y = 0$

$\dot{y} + 2x - y = 0$

$x = Ae^{\lambda t}, \ y = Be^{\lambda t}$

$A\lambda - 4A - B = 0$

$B\lambda + 2A - B = 0$

$A(\lambda - 4) - B = 0$

$2A + B(\lambda - 1) = 0$

$$\begin{vmatrix} r-4 & -1 \\ 2 & r-1 \end{vmatrix} = 0$$

$(r-4)(r-1)+2=0$

$r^2-4r-r+4+2=0$

$r^2-5r+6=0$

$r_1=2 \qquad r_2=3$

$\underline{r=2:}$ $\quad -2A-B=0$

$\qquad 2A+B=0$

$B=-2A \qquad \underline{A=1, \quad B=-2}$

$$\boxed{x_1=e^{2t}, \quad y_1=-2e^{2t}}$$

$\underline{r=3:}$ $\quad -A-B=0 \quad / \cdot(-1)$

$\qquad 2A+2B=0 \quad / : 2$

$\qquad A+B=0$

$\qquad A=1, \quad B=-1$

$x_2=e^{3t}, \quad y_2=-e^{3t}$

$x_h=C_1e^{2t}+C_2e^{3t}$

$y_h=-2C_1e^{2t}-C_2e^{3t}$

Tražimo opće rj. polaznog sis. u obliku:

$x=C_1(t)e^{2t}+C_2(t)e^{3t}$

$y=-2C_1(t)e^{2t}-C_2(t)e^{3t}$

pa uvrstimo u polazni sis.:

$$C_1' e^{2t} + \cancel{C_1 e^{2t}\cdot 2} + C_2' e^{3t} + \cancel{C_2 e^{3t}\cdot 3} - \cancel{4C_1 e^{2t}} - \cancel{4C_2 e^{3t}} + \cancel{2C_1 e^{2t}} + \cancel{C_2 e^{3t}} = -36t$$

$$-2\left(C_1' e^{2t} + \cancel{C_1 e^{2t}\cdot 2}\right) - \left(C_2' e^{3t} + \cancel{C_2 e^{3t}\cdot 3}\right) + \cancel{2C_1 e^{2t}} + \cancel{2C_2 e^{3t}} + \cancel{2C_1 e^{2t}} + \cancel{C_2 e^{3t}} = -2e^t$$

$$\left.\begin{aligned} C_1' e^{2t} + C_2' e^{3t} &= -36t \\ -2C_1' e^{2t} - C_2' e^{3t} &= -2e^t \end{aligned}\right\} +$$

$$-C_1' e^{2t} = -36t - 2e^t \qquad /\cdot\left(-e^{-2t}\right)$$

$$C_1' = \left(36t + 2e^t\right) e^{-2t}$$

$$C_1' = 36te^{-2t} + 2e^{-t}$$

$$C_2' e^{3t} = -36t - 36t - 2e^t \qquad / e^{-3t}$$

$$C_2' = -72te^{-3t} - 2e^{-2t}$$

$$C_1(t) = \int\left(36te^{-2t} + 2e^{-t}\right)dt = \dots = -2e^{-t} - 18te^{-2t} - 9e^{-2t} + D_1$$

$$C_2(t) = \int\left(-72te^{-3t} - 2e^{-2t}\right)dt = \dots = e^{-2t} + 24te^{-3t} + 8e^{-3t} + D_2$$

Uvrštavanjem dobijemo:

$$x = D_1 e^{2t} + D_2 e^{3t} - e^t + 6t - 1$$

$$y = -2D_1 e^{2t} - D_2 e^{3t} + 12t + 3e^t + 10$$

Za vježbu: (neki su bili na ispitu)

2. $\dot{x} - 3x + 4y = e^{-2t}$

   $\dot{y} - x + 2y = -3e^{2t}$

3. $\dot{x} = y + 2e^t$

   $\dot{y} = x + t^2$

4. $\dot{x} = -4x - 2y + \dfrac{2}{e^t - 1}$

   $\dot{y} = 6x + 3y - \dfrac{3}{e^t - 1}$

5. $\dot{x} = y + \operatorname{tg}^2 t - 1$

   $\dot{y} = -x + \operatorname{tg} t$

NEHOMOGENI SISTEMI – METODOM NALAŽENJE PARTIKULARNOG RJEŠENJA

1. $\dot{x} = y - 5\cos t$

$\dot{y} = 2x + y$

$\dot{x} = y$

$\dot{y} = 2x + y$

$x = Ae^{rt}, \quad y = Be^{rt}$

$Ar = B$

$Br = 2A + B$

$Ar - B = 0$

$-2A + B(r-1) = 0$

k. jed. $\begin{vmatrix} r & -1 \\ -2 & r-1 \end{vmatrix} = 0$

$r^2 - r - 2 = 0 \Rightarrow r_1 = 2, \; r_2 = -1$

$r = 2: \quad 2A - B = 0$

$-2A + B = 0 \quad \Rightarrow B = 2A$

$A = 1, \; B = 2$

$x_1 = e^{2t} \qquad y_1 = 2e^{2t}$

$r = -1 \quad -A - B = 0 \quad / \cdot (-1)$

$-2A - 2B = 0$

$A + B = 0 \Rightarrow B = -A$

$A = 1, \; B = -1$

$x_2 = e^{-t} \qquad y_2 = -e^{-t}$

$x_h = C_1 e^{2t} + C_2 e^{-t}$

$y_h = 2C_1 e^{2t} - C_2 e^{-t}$

Rješenje nehomogenog sis.: $\begin{cases} x = x_h + x_p \\ y = y_h + y_p \end{cases}$

Partikularno rješenje:

$$\begin{aligned} x_p &= a\cos t + b\sin t \\ y_p &= c\cos t + d\sin t \end{aligned} \Rightarrow \begin{aligned} \dot{x}_p &= -a\sin t + b\cos t \\ \dot{y}_p &= -c\sin t + d\cos t \end{aligned}$$

$-a\sin t + b\cos t = c\cos t + d\sin t - 5\cos t$

$-c\sin t + d\cos t = 2a\cos t + 2b\sin t + c\cdot\cos t + d\cdot\sin t$

uz $\cos t$: $\begin{cases} b = c - 5 \\ d = 2a + c \end{cases}$ uz $\sin t$: $\begin{cases} -a = d \\ -c = 2b + d \end{cases}$

$d = -a \Rightarrow -a = 2a + c \Rightarrow c = -3a$

$b = c - 5$

$b = -3a - 5$

$-c = 2b + d$

$3a = -6a - 10 - a$

$3a + 6a + a = -10$

$10a = -10 \Rightarrow a = -1$

$b = -2, \; c = 3, \; d = 1$

$x_p = -\cos t - 2\sin t$

$y_p = 3\cos t + \sin t$

Rješenje sis.: $\begin{cases} x = C_1 e^{2t} + C_2 e^{-t} - \cos t - 2\sin t \\ y = 2C_1 e^{2t} - C_2 e^{-t} + 3\cos t + \sin t \end{cases}$

2. $\dot{x} = 2x + y - 7te^{-t} - 3$

$\dot{y} = -x + 2y - 1$

$$\left.\begin{aligned} \dot{x} &= 2x + y \\ \dot{y} &= -x + 2y \end{aligned}\right\} \text{hom. sis.}$$

$x = Ae^{\lambda t}, \quad y = Be^{\lambda t}$

$A\lambda = 2A + B$

$B\lambda = -A + 2B$

$A(\lambda - 2) - B = 0$

$A + B(\lambda - 2) = 0$

k.j. $\begin{vmatrix} \lambda - 2 & -1 \\ 1 & \lambda - 2 \end{vmatrix} = 0$

$(\lambda - 2)^2 + 1 = 0$

$(\lambda - 2)^2 = -1$

$\lambda - 2 = \pm i \Rightarrow \lambda_{1,2} = 2 \pm i$

$\lambda = 2 + i$:

$$\left.\begin{aligned} iA - B &= 0 \\ A + iB &= 0 \end{aligned}\right\} \Leftrightarrow \quad / \cdot i \Rightarrow iA - B = 0$$

$iA = B$

$A = 1, \quad B = i$

$\overline{x}_1 = e^{(2+i)t} = e^{2t} \cdot e^{it} = e^{2t} \cdot (\cos t + i \sin t)$

$\overline{y}_1 = i e^{(2+i)t} = e^{2t} \cdot (-\sin t + i \cos t)$

$\lambda = 2 - i$: $\quad -iA - B = 0$, $\quad A - iB = 0$ $\Leftrightarrow$

$-iA = B \Rightarrow A = 1, B = -i$

$$\overline{x_2} = e^{(2-i)t} = e^{2t} \cdot (\cos t - i \sin t)$$
$$\overline{y_2} = -i\, e^{(2-i)t} = e^{2t} (-\sin t - i\cos t)$$

$$\overline{x_1} + \overline{x_2} = 2e^{2t} \cdot \cos t \qquad \overline{y_1} + \overline{y_2} = -2e^{2t} \sin t$$
$$\overline{x_1} - \overline{x_2} = 2ie^{2t} \sin t \qquad \overline{y_1} - \overline{y_2} = 2ie^{2t} \cos t$$

$$x_1 = \frac{\overline{x_1} + \overline{x_2}}{2} = e^{2t} \cos t \qquad y_1 = \frac{\overline{y_1} + \overline{y_2}}{2} = -e^{2t} \sin t$$

$$x_2 = \frac{\overline{x_1} - \overline{x_2}}{2i} = e^{2t} \sin t \qquad y_2 = \frac{\overline{y_1} - \overline{y_2}}{2i} = e^{2t} \cos t$$

$$x_h = e^{2t} (c_1 \cos t + c_2 \sin t)$$

$$y_h = e^{2t} (-c_1 \sin t + c_2 \cos t)$$

$$\left.\begin{aligned} x_p &= (at + b)e^{-t} + c \\ y_p &= (dt + f)e^{-t} + g \end{aligned}\right\}$$ slobodne članove kombiniramo; moraju biti istog oblika - uvijek;

$$\dot{x}_p = ae^{-t} + (at + b)e^{-t} \cdot (-1)$$
$$\dot{y}_p = de^{-t} + (dt + f)e^{-t} (-1)$$

$$x \to e^{-t}(a - at - b) = 2 \cdot (at + b)e^{-t} + 2c + (dt + f)e^{-t} + g - 7te^{-t} - 3$$
$$y \to e^{-t}(d - dt - f) = -(at + b)e^{-t} - c + 2 \cdot (dt + f)e^{-t} + 2g - 1$$

$$e^{-t}(a - at - b - 2at - 2b - dt - f + 7t) = 2c + g - 3$$
$$e^{-t}(d - dt - f + at + b - 2dt - 2f) = -c + 2g - 1$$

$$e^{t}\left(\underbrace{-3at - dt + 7t}_{0} + \underbrace{a - 3b - f}_{0}\right) = \underbrace{2c + g - 3}_{0}$$

$$e^{t}\left(\underbrace{at - 3dt}_{0} + \underbrace{b + d - 3f}_{0}\right) = \underbrace{-c + 2g - 1}_{0}$$

zbog lin. nezavisnosti $\Rightarrow$ $=0$

$-3a - d + 7 = 0$

$a - 3b - f = 0$

$2c + g - 3 = 0$

$a - 3d = 0 \Rightarrow a = 3d$

$b + d - 3f = 0$

$-c + 2g - 1 = 0$

$-9d - d + 7 = 0$

$-10d = -7$

$\boxed{d = \frac{7}{10}}$

$\boxed{a = \frac{21}{10}}$

$a - 3b - f = 0$

$b + d - 3f = 0$

$\frac{21}{10} = 3b + f \quad / \cdot 3$

$-\frac{7}{10} = b - 3f$

$\left.\begin{array}{l} \frac{63}{10} = 9b + 3f \\ -\frac{7}{10} = b - 3f \end{array}\right\} +$

$\frac{56}{10} = 10b \Rightarrow b = \frac{56}{100} = \frac{14}{25}$

$f = \frac{21}{10} - 3 \cdot \frac{14}{25} = \frac{105 - 84}{50} = \frac{21}{50}$

$2c + g = 3$

$-c + 2g = 1 \quad / \cdot 2$

$2c + g = 3$

$-2c + 4g = 2$

$5g = 5$

$g = 1$, $2c = 2 \Rightarrow c = 1$

Konačno rj:

$\Rightarrow \quad x = e^{2t}(c_1 \cos t + c_2 \sin t) + \left(\frac{21}{10}t + \frac{14}{25}\right)e^{-t} + 1$

$y = e^{2t}(c_1 \sin t + c_2 \cos t) + \left(\frac{7}{10}t + \frac{21}{50}\right)e^{-t} + 1$

za vježbu:

3. $\dot{x} = 2x + y + 2e^{t}$

$\dot{y} = x + 2y - 3e^{4t}$

problem

Uputa: $\lambda_1 = 1$, $\lambda_2 = 3$;

$x_p = ate^{t} + be^{4t}$

$y_p = cte^{t} + de^{4t}$

4. $\dot{x} = x - y + 8t$

$\dot{y} = 5x - y$

5. $\dot{x} = 2x - y$

$\dot{y} = 2y - x - 5 \cdot e^{t} \sin t$

6. $\dot{x} = x + 2y + 16te^{t}$

$\dot{y} = 2x - 2y$